Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity, II

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Abstract

We present two constraint minimization approaches to prove the existence of traveling waves for a wide class of nonlinear Schrödinger equations with nonvanishing conditions at infinity in space dimension $N \geq 2$. Minimization of the energy at fixed momentum can be used whenever the associated nonlinear potential is nonnegative and it gives a set of orbitally stable traveling waves. Minimization of the action at constant kinetic energy can be used in all cases, but it gives no information about the orbital stability of the set of solutions.

Keywords. nonlinear Schrödinger equation, nonzero conditions at infinity, traveling wave, Gross-Pitaevskii equation, cubic-quintic NLS, constrained minimization, Ginzburg-Landau energy.

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1 Introduction

We study a class of special solutions to the nonlinear Schrödinger equation

$$i\frac{\partial \Phi}{\partial t} + \Delta \Phi + F(|\Phi|^2)\Phi = 0 \qquad \text{in } \mathbf{R}^N, \tag{1.1}$$

where Φ is a complex-valued function on $\mathbf{R}^N$ satisfying the "boundary condition" $|\Phi| \longrightarrow r_0$ as $|x| \longrightarrow \infty$, $r_0 > 0$ and F is a real-valued function on $\mathbf{R}_+$ such that $F(r_0^2) = 0$.

Equation (1.1), with the considered non-zero conditions at infinity, arises in the modeling of a great variety of physical phenomena such as superconductivity, superfluidity in Helium II, phase transitions and Bose-Einstein condensate ([1], [3], [4], [5], [18], [27], [29], [30], [31], [32], [44]). In nonlinear optics, it appears in the context of dark solitons ([35], [36]). Two important model cases for (1.1) have been extensively studied both in the physical and mathematical literature: the Gross-Pitaevskii equation (where $F(s) = 1-s$) and the so-called "cubic-quintic" Schrödinger equation (where $F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, α_1, α_3, α_5 are positive and F has two positive roots).

In contrast to the case of zero boundary conditions at infinity (when the dynamics associated to (1.1) is essentially governed by dispersion and scattering), the non-zero boundary conditions allow a much richer dynamics and give rise to a remarkable variety of special solutions, such as traveling waves, standing waves or vortex solutions.

Using the Madelung transformation $\Phi(x,t) = \sqrt{\rho(x,t)}e^{i\theta(x,t)}$ (which is well-defined in any region where $\Phi \neq 0$), equation (1.1) is equivalent to a system of Euler's equations for a compressible inviscid fluid of density ρ and velocity $2\nabla\theta$. In this context it has been shown that, if F is C^1 near r_0^2 and $F'(r_0^2) < 0$, the sound velocity at infinity associated to (1.1) is $v_s = r_0\sqrt{-2F'(r_0^2)}$ (see the introduction of [41]).

If $F'(r_0^2) < 0$ (which means that (1.1) is defocusing), a simple scaling enables us to assume that $r_0 = 1$ and $F'(r_0^2) = -1$; we will do so throughout the rest of this paper. The sound velocity at infinity is then $v_s = \sqrt{2}$.

Equation (1.1) has a Hamiltonian structure. Indeed, let $V(s) = \int_s^1 F(\tau)\,d\tau$. It is then easy to see that, at least formally, the "energy"

$$E(\Phi) = \int_{\mathbf{R}^N} |\nabla \Phi|^2\,dx + \int_{\mathbf{R}^N} V(|\Phi|^2)\,dx \tag{1.2}$$

is conserved. Another quantity which is conserved by the flow of (1.1) is the momentum, $\mathbf{P}(\Phi) = (P_1(\Phi), \dots, P_N(\Phi))$. A rigorous definition of the momentum will be given in the next section. If Φ is a function sufficiently localized in space, we have $P_k(\Phi) = \int_{\mathbf{R}^N} \langle i\Phi_{x_k}, \Phi\rangle\,dx$, where $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbf{C} \simeq \mathbf{R}^2$.

In a series of papers (see, e.g., [3], [4], [27], [31], [32]), particular attention has been paid to the traveling waves of (1.1). These are solutions of the form $\Phi(x,t) = \psi(x + ct\omega)$, where $\omega \in S^{N-1}$ is the direction of propagation and $c \in \mathbf{R}^*$ is the speed of the traveling wave. They are supposed to play an important role in the dynamics of (1.1). We say that ψ has finite energy if $\nabla\psi \in L^2(\mathbf{R}^N)$ and $V(|\psi|^2) \in L^1(\mathbf{R}^N)$. Since the equation (1.1) is rotation invariant, we may assume that $\omega = (1, 0, \dots, 0)$. Then a traveling wave of speed c satisfies the equation

$$ic\frac{\partial \psi}{\partial x_1} + \Delta\psi + F(|\psi|^2)\psi = 0 \qquad \text{in } \mathbf{R}^N. \tag{1.3}$$

It is obvious that a function ψ satisfies (1.3) for some velocity c if and only if $\psi(-x_1, x')$ satisfies (1.3) with c replaced by $-c$. Hence it suffices to consider the case $c \geq 0$.

In view of formal computations and numerical experiments, it has been conjectured that finite energy traveling waves of speed c exist only for subsonic speeds: $c < v_s$. The nonexistence of traveling waves for supersonic speeds ($c > v_s$) has been proven first in [28] in the case of the Gross-Pitaevskii equation, then in [41] for a wide class of nonlinearities. More qualitatively, the numerical investigation of the traveling waves of the Gross-Pitaevskii equation ($F(s) = 1 - s$) has been carried out in [31]. The method used there was a continuation argument with respect to the speed, solving (1.3) by Newton's algorithm. Denoting $Q(\psi) = P_1(\psi)$ the momentum of ψ with respect to the x_1-direction, the representation of solutions in the energy vs. momentum diagram gives the following curves (the straight line is the line $E = v_s Q$).

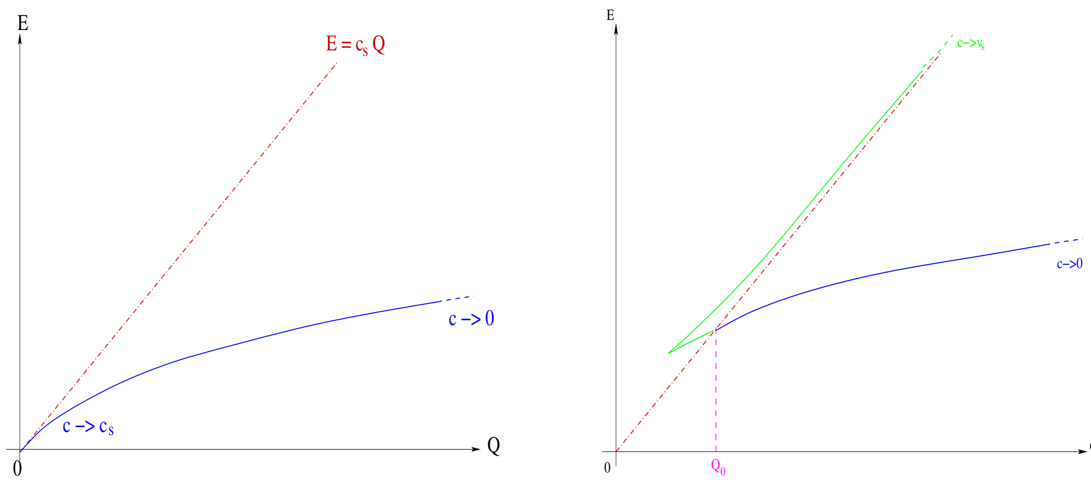

Figure 1: (E, P) diagrams for (GP): (a) dimension $N = 2$; (b) dimension $N = 3$.

The rigorous proof of the existence of traveling waves has been a long lasting problem and was considered in a series of papers, see [9], [8], [14], [7], [43]. At least formally, traveling waves are critical points of the functional $E - cQ$. Therefore, it is a natural idea to look for such solutions as minimizers of the energy at fixed momentum, the speed c being then the Lagrange multiplier associated to the minimization problem. In the case of the Gross-Pitaevskii equation, in view of the above diagrams, this method is expected to give the full curve of traveling waves if $N = 2$ and only the lower part that lies under the line $E = v_s Q$ if $N = 3$ (because it is clear that minimizers of E at fixed Q cannot lie on the upper branch). On a rigorous level, minimizing the energy at fixed momentum was used in [8] to construct a sequence of traveling waves with speeds $c_n \longrightarrow 0$ in dimension $N \geq 3$. Minimizing the energy E at fixed momentum Q has the advantage of providing orbitally stable traveling waves, and this is intimately related to the concavity of the curve $Q \mapsto E$. On the other hand, if $Q \mapsto E$ is convex, as it is the case on the upper branch in figure 1 (b), one expects orbital instability.

More recently, the curves describing the minimum of the energy at fixed momentum in dimension 2 and 3 have been obtained in [7], where the existence of minimizers of E under the constraint $Q = constant$ is also proven for any $q > 0$ if $N = 2$, respectively for any $q \in (q_0, \infty)$ (with $q_0 > 0$) if $N = 3$. The proofs in [7] depend on the special algebraic structure of the Gross-Pitaevskii nonlinearity and it seems difficult to extend them to other nonlinearities. The existence of minimizers has been shown by considering the corresponding problem on tori $(\mathbf{R}/2n\pi\mathbf{Z})^N$, proving a priori bounds for minimizers on tori, then passing to the limit as $n \longrightarrow \infty$. Although this method gives the existence of minimizers on $\mathbf{R}^N$, it does not imply the precompactness of all minimizing sequences, and therefore leaves the question of the orbital stability of minimizers completely open.

The existence of traveling waves for (1.1) under general conditions on the nonlinearity, in any space dimension $N \geq 3$ and for any speed $c \in (0, v_s)$ has been proven in [43] by minimizing the action $E - cQ$ under a Pohozaev constraint. The method in [43] cannot be used in space dimension two (there are no minimizers under Pohozaev constraints). Although the traveling waves obtained in [43] minimize the action $E - cQ$ among all traveling waves of speed c, the constraint used to prove their existence is not conserved by the flow of (1.1) and consequently it seems very difficult to prove their orbital stability (which is expected at least for small speeds c).

In the present paper we adopt a different strategy. If the nonlinear potential V is nonnegative, we consider the problem of minimizing the energy at fixed momentum $Q = q$ and we show that in any space dimension $N \geq 2$ there exist minimizers for any $q \in (q_0, \infty)$, with $q_0 \geq 0$. The minimizers are traveling waves and their speeds are the Lagrange multipliers associated to the variational problem. These speeds tend to zero as $q \longrightarrow \infty$. If $N = 2$ and F has a good behavior near 1 (more precisely, if assumption (A4) below is satisfied and the "nondegeneracy condition" $F''(1) \neq 3$ holds), we prove that $q_0 = 0$ and that the speeds of the traveling waves that we obtain tend to v_s as $q \longrightarrow 0$. For general nonlinearities we obtain the properties of the minimum of the energy vs. momentum curve and this is in agreement with the results in [31], [32] and [7]. We also prove the precompactness of all minimizing sequences for the above mentioned problem, which implies the orbital stability of the set of traveling waves obtained in this way.

If V achieves negative values (this happens, for instance, in the case of the cubic-quintic NLS), the infimum of the energy in the set of functions of constant momentum is always $-\infty$. In this case we minimize the functional $E - Q$ in the set of functions ψ satisfying $\int_{\mathbf{R}^N} |\nabla \psi|^2 \, dx = k$. In space dimension $N \geq 2$ we prove that minimizers exist for any k in some interval (k_0, k_∞) and, after scaling, they give rise to traveling waves. Moreover, if $N = 2$ and F behaves nicely near 1 we have $k_0 = 0$ and the speeds of traveling waves obtained in this way tend to v_s as $k \longrightarrow 0$. Let us emphasize that the result of [43], which holds for any $N \geq 3$, does not require any sign assumption on the potential V.

In space dimension two, even if V takes negative values it is still possible to find local minimizers of the energy under the constraint $Q = q = \mathit{constant}$ if q is not too large. If F satisfies assumption (A4) below and $F''(1) \neq 3$ this can be done for any q in some interval $(0, q_\infty)$ and the speeds of traveling waves obtained in this way tend to v_s as $q \longrightarrow 0$. Moreover, we get the precompactness of all minimizing sequences, and consequently the orbital stability of the set of local minimizers.

Our results cover as well nonlinearities of Gross-Pitaevskii type and of cubic-quintic type. To the best of our knowledge, all previous results in the literature about the existence of traveling waves for (1.1) in space dimension two are concerned only with the Gross-Pitaevskii equation and the proofs make use of the specific algebraic properties of this nonlinearity.

The main disadvantage of the present approaches is that although we get minimizers for any momentum in some interval (q_0, ∞) or $(0, q_\infty)$ or for any kinetic energy in some interval (k_0, k_∞), the speeds of the traveling waves obtained in this way are Lagrange multipliers, so we cannot guarantee that these speeds cover a whole interval. However, in all cases it can be proved that we get an uncountable set of speeds.

One might ask whether there is a relationship between the families of traveling waves obtained from different minimization problems. In dimension $N \geq 3$ we prove that all traveling waves found in the present paper also minimize the action $E - cQ$ under the Pohozaev constraint considered in [43]. The converse is, in general, not true. For instance, in the case of the Gross-Pitaevskii equation in dimension $N \geq 3$, it was proved in [7, 21] that there are no traveling waves of small energy, and we generalize that result in the present paper; this implies that there is $c_0 < v_s$ such that there are no traveling waves of speed $c \in (c_0, v_s)$ which minimize the energy at fixed momentum. However, if $N \geq 3$ the existence of traveling waves as minimizers of $E - cQ$ under a Pohozaev constraint has been proven for any $c \in (0, v_s)$. This is in agreement with the energy-momentum diagram of figure 1 (b), where the traveling waves with speed c close to the speed of sound v_s are expected to be on the upper branch. We also prove that all minimizers of the energy at fixed momentum are (after scaling) minimizers of $E - Q$ at fixed kinetic energy. It is an open question whether the converse is true or not. An affirmative answer to this question would imply that the set of speeds of traveling waves which minimize the energy at fixed momentum is an interval. However, this last fact might not be true, at least if we do not impose further conditions on the nonlinearity F. Indeed, in the case of general nonlinearities (as those studied in dimension one in [15]), the two-dimensional traveling waves to (1.1) have been studied numerically in [17]. The numerical algorithms in [17] allow to perform the constrained minimization procedures used is the present paper. It appears that for $N = 2$, even if the potential V is nonnegative, it is not true in general that minimizing E at fixed Q or minimizing $E - Q$ at fixed kinetic energy provides a single interval of speeds; for instance, it may provide the union of two disjoint intervals.

We will consider the following set of assumptions:

(A1) The function F is continuous on $[0, \infty)$, C^1 in a neighborhood of 1, $F(1) = 0$ and $F'(1) < 0$.

(A2) There exist $C > 0$ and $p_0 < \frac{2}{N-2}$ (with $p_0 < \infty$ if $N = 2$) such that $|F(s)| \leq C(1 + s^{p_0})$ for any $s \geq 0$.

(A3) There exist C, $\alpha_0 > 0$ and $r_* > 1$ such that $F(s) \leq -Cs^{\alpha_0}$ for any $s \geq r_*$.

(A4) F is C^2 near 1 and

$$F(s) = -(s-1) + \frac{1}{2}F''(1)(s-1)^2 + \mathcal{O}((s-1)^3) \qquad \text{for } s \text{ close to } 1.$$

If (A1) and (A3) are satisfied, it is explained in the introduction of [43] how it is possible to modify F is a neighborhood of infinity in such a way that the modified function $\tilde{F}$ satisfies also (A2) and (1.1) has the same traveling waves as the equation obtained from it by replacing F with $\tilde{F}$. If (A1) and (A2) hold, we get traveling waves as minimizers of some functionals under constraints. However, if (A1) and (A3) are verified but (A2) is not, the above argument implies only the existence of such solutions, and not the fact that they are minimizers.

If F satisfies (A1), using Taylor's formula for s in a neighborhood of 1 we have

$$V(s) = \frac{1}{2}V''(1)(s-1)^2 + (s-1)^2\varepsilon(s-1) = \frac{1}{2}(s-1)^2 + (s-1)^2\varepsilon(s-1), \tag{1.4}$$

where $\varepsilon(t) \longrightarrow 0$ as $t \longrightarrow 0$. Hence for $|\psi|$ close to 1, $V(|\psi|^2)$ can be approximated by the Ginzburg-Landau potential $\frac{1}{2}(|\psi|^2 - 1)^2$.

Energy and function spaces. We fix an odd function $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi(s) = s$ for $s \in [0, 2]$, $0 \leq \varphi' \leq 1$ on $\mathbf{R}$ and $\varphi(s) = 3$ for $s \geq 4$. If assumptions (A1) and (A2) are satisfied, it is not hard to see that there exist C_1, C_2, $C_3 > 0$ such that

$$\begin{aligned} &|V(s)| \leq C_1(s-1)^2 \quad \text{for any } s \leq 9; \\ &\text{in particular, } |V(\varphi^2(\tau))| \leq C_1(\varphi^2(\tau) - 1)^2 \text{ for any } \tau, \end{aligned} \tag{1.5}$$

$$|V(b) - V(a)| \leq C_2|b - a| \max(a^{p_0}, b^{p_0}) \qquad \text{for any } a,\, b \geq 2. \tag{1.6}$$

Given $\psi \in H^1_{loc}(\mathbf{R}^N)$ and an open set $\Omega \subset \mathbf{R}^N$, the modified Ginzburg-Landau energy of ψ in Ω is defined by

$$E^{\Omega}_{GL}(\psi) = \int_\Omega |\nabla\psi|^2\, dx + \frac{1}{2}\int_\Omega \left(\varphi^2(|\psi|) - 1\right)^2\, dx. \tag{1.7}$$

We simply write $E_{GL}(\psi)$ instead of $E_{GL}^{\mathbf{R}^N}(\psi)$. The modified Ginzburg-Landau energy will play a central role in our analysis.

We denote $\dot{H}^1(\mathbf{R}^N) = \{\psi \in L^1_{loc}(\mathbf{R}^N) \mid \nabla\psi \in L^2(\mathbf{R}^N)\}$ and

$$(1.8) \qquad \begin{aligned} \mathcal{E} &= \{\psi \in \dot{H}^1(\mathbf{R}^N) \bigm| \varphi^2(|\psi|) - 1 \in L^2(\mathbf{R}^N)\} \\ &= \{\psi \in \dot{H}^1(\mathbf{R}^N) \bigm| E_{GL}(\psi) < \infty\}. \end{aligned}$$

Let $\mathcal{D}^{1,2}(\mathbf{R}^N)$ be the completion of $C_c^\infty(\mathbf{R}^N)$ for the norm $\|v\| = \|\nabla v\|_{L^2(\mathbf{R}^N)}$ and let

$$(1.9) \qquad \begin{aligned} \mathcal{X} &= \{u \in \mathcal{D}^{1,2}(\mathbf{R}^N) \bigm| \varphi^2(|1+u|) - 1 \in L^2(\mathbf{R}^N)\} \\ &= \{u \in \dot{H}^1(\mathbf{R}^N) \bigm| u \in L^{2^*}(\mathbf{R}^N),\ E_{GL}(1+u) < \infty\} \quad \text{if } N \geq 3. \end{aligned}$$

If $N \geq 3$ and $\psi \in \mathcal{E}$, there exists a constant $z_0 \in \mathbf{C}$ such that $\psi - z_0 \in L^{2^*}(\mathbf{R}^N)$, where $2^* = \frac{2N}{N-2}$ (see, for instance, Lemma 7 and Remark 4.2 pp. 774-775 in [24]). It follows that $\varphi(|\psi|) - \varphi(|z_0|) \in L^{2^*}(\mathbf{R}^N)$. On the other hand, the fact that $E_{GL}(\psi) < \infty$ implies $\varphi(|\psi|) - 1 \in L^2(\mathbf{R}^N)$, thus necessarily $\varphi(|z_0|) = 1$, that is $|z_0| = 1$. Then it is easily seen that there exist $\alpha_0 \in [0, 2\pi)$ and $u \in \mathcal{X}$, uniquely determined by ψ, such that $\psi = e^{i\alpha_0}(1+u)$. In other words, if $N \geq 3$ we have $\mathcal{E} = \{e^{i\alpha_0}(1+u) \mid \alpha_0 \in [0,2\pi),\ u \in \mathcal{X}\}$.

It is not hard to see that for $N \geq 2$ we have

$$(1.10) \qquad \mathcal{E} = \{\psi : \mathbf{R}^N \longrightarrow \mathbf{C} \bigm| \psi \text{ is measurable}, |\psi| - 1 \in L^2(\mathbf{R}^N), \nabla\psi \in L^2(\mathbf{R}^N)\}.$$

Indeed, we have $\big|\varphi^2(|\psi|) - 1\big| \leq 4\big|\,|\psi| - 1\big|$, hence $\varphi^2(|\psi|) - 1 \in L^2(\mathbf{R}^N)$ if $|\psi| - 1 \in L^2(\mathbf{R}^N)$. Conversely, let $\psi \in \mathcal{E}$. If $N = 2$, it follows from Lemma 2.1 below that $|\psi|^2 - 1 \in L^2(\mathbf{R}^2)$ and we have $\big|\,|\psi| - 1\big| = \frac{1}{|\psi|+1}\big|\,|\psi|^2 - 1\big| \leq \big|\,|\psi|^2 - 1\big|$. If $N \geq 3$, we know that $\varphi(|\psi|) - 1 \in L^2(\mathbf{R}^N)$ and $0 \leq |\psi| - \varphi(|\psi|) \leq |\psi| \mathbb{1}_{\{|\psi| \geq 2\}} \leq 2(|\psi| - 1)\mathbb{1}_{\{|\psi| \geq 2\}} \leq 2\big|\,|\psi| - 1\big|^{\frac{2^*}{2}} \mathbb{1}_{\{|\psi| \geq 2\}}$ and the last function belongs to $L^2(\mathbf{R}^N)$ by the Sobolev embedding. Moreover, one may find bounds for $\|\,|\psi| - 1\|_{L^2(\mathbf{R}^N)}$ in terms of $E_{GL}(\psi)$ (see Corollary 4.3 below).

Proceeding as in [25], section 1, one proves that $\mathcal{E} \subset L^2 + L^\infty(\mathbf{R}^N)$ and that $\mathcal{E}$ endowed with the distance

$$(1.11) \qquad d_{\mathcal{E}}(\psi_1, \psi_2) = \|\psi_1 - \psi_2\|_{L^2 + L^\infty(\mathbf{R}^N)} + \|\nabla\psi_1 - \nabla\psi_2\|_{L^2(\mathbf{R}^N)} + \|\,|\psi_1| - |\psi_2|\,\|_{L^2(\mathbf{R}^N)}$$

is a complete metric space. We recall that, given two Banach spaces X and Y of distributions on $\mathbf{R}^N$, the space $X + Y$ with norm defined by $\|w\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y \bigm| w = x + y, x \in X, y \in Y\}$ is a Banach space.

We will also consider the following semi-distance on $\mathcal{E}$:

$$(1.12) \qquad d_0(\psi_1, \psi_2) = \|\nabla\psi_1 - \nabla\psi_2\|_{L^2(\mathbf{R}^N)} + \|\,|\psi_1| - |\psi_2|\,\|_{L^2(\mathbf{R}^N)}.$$

If $\psi_1, \psi_2 \in \mathcal{E}$ and $d_0(\psi_1, \psi_2) = 0$, then we have $|\psi_1| = |\psi_2|$ a.e. on $\mathbf{R}^N$ and $\psi_1 - \psi_2$ is a constant (of modulus not exceeding 2) a.e. on $\mathbf{R}^N$.

In space dimension $N = 2, 3, 4$, the Cauchy problem for the Gross-Pitaevskii equation has been studied by Patrick Gérard ([24, 25]) in the space naturally associated to that equation, namely

$$\mathbf{E} = \{\psi \in H^1_{loc}(\mathbf{R}^N) \mid \nabla\psi \in L^2(\mathbf{R}^N), |\psi|^2 - 1 \in L^2(\mathbf{R}^N)\}$$

endowed with the distance

$$d_{\mathbf{E}}(\psi_1, \psi_2) = \|\psi_1 - \psi_2\|_{L^2 + L^\infty(\mathbf{R}^N)} + \|\nabla\psi_1 - \nabla\psi_2\|_{L^2(\mathbf{R}^N)} + \|\,|\psi_1|^2 - |\psi_2|^2\,\|_{L^2(\mathbf{R}^N)}.$$

If $N = 2, 3$ or 4 it can be proven that $\mathbf{E} = \mathcal{E}$ and the distances $d_{\mathcal{E}}$ and $d_{\mathbf{E}}$ are equivalent on $\mathcal{E}$. Global well-posedness was shown in [24, 25] (see section 7) if $N \in \{2, 3\}$ or if $N = 4$ and the initial data is small. In the case $N = 4$, global well-posedness for any initial data in $\mathbf{E}$ was recently proven in [34].

Notation. Throughout the paper, $\mathcal{L}^N$ is the Lebesgue measure on $\mathbf{R}^N$ and $\mathcal{H}^s$ is the $s-$dimensional Hausdorff measure on $\mathbf{R}^N$. For $x = (x_1, \dots, x_N) \in \mathbf{R}^N$, we denote $x' = (x_2, \dots, x_N) \in \mathbf{R}^{N-1}$. We write $\langle z_1, z_2 \rangle$ for the scalar product of two complex numbers z_1, z_2. Given a function f defined on $\mathbf{R}^N$ and λ, $\sigma > 0$, we denote

$$f_{\lambda,\sigma}(x) = f\left(\frac{x_1}{\lambda}, \frac{x'}{\sigma}\right). \tag{1.13}$$

If $1 \le p < N$, we write p^* for the Sobolev exponent associated to p, that is $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

Main results. Our most important results can be summarized as follows.

Theorem 1.1 *Assume that $N \ge 2$, (A1) and (A2) are satisfied and $V \ge 0$ on $[0, \infty)$. For $q \ge 0$, let*

$$E_{min}(q) = \inf\{E(\psi) \mid \psi \in \mathcal{E},\ Q(\psi) = q\}.$$

Then:

(i) The function E_{min} is concave, increasing on $[0, \infty)$, $E_{min}(q) \le v_s q$ for any $q \ge 0$, the right derivative of E_{min} at 0 is v_s, and $E_{min}(q) \longrightarrow \infty$ and $\frac{E_{min}(q)}{q} \longrightarrow 0$ as $q \longrightarrow \infty$.

(ii) Let $q_0 = \inf\{q > 0 \mid E_{min}(q) < v_s q\}$. For any $q > q_0$, all sequences $(\psi_n)_{n\ge1} \subset \mathcal{E}$ satisfying $Q(\psi_n) \longrightarrow q$ and $E(\psi_n) \longrightarrow E_{min}(q)$ are precompact for d_0 (modulo translations).
The set $\mathcal{S}_q = \{\psi \in \mathcal{E} \mid Q(\psi) = q,\ E(\psi) = E_{min}(q)\}$ is not empty and is orbitally stable (for the semi-distance d_0) by the flow associated to (1.1).

(iii) Any $\psi_q \in \mathcal{S}_q$ is a traveling wave for (1.1) of speed $c(\psi_q) \in [d^+E_{min}(q), d^-E_{min}(q)]$, where we denote by d^- and d^+ the left and right derivatives. We have $c(\psi_q) \longrightarrow 0$ as $q \longrightarrow \infty$.

(iv) If $N \ge 3$ we have always $q_0 > 0$. Moreover, if $N = 2$ and assumption (A4) is satisfied, we have $q_0 = 0$ if and only if $F''(1) \ne 3$, in which case $c(\psi_q) \longrightarrow v_s$ as $q \longrightarrow 0$.

If V achieves negative values, the infimum of E on the set $\{\psi \in \mathcal{E} \mid Q(\psi) = q\}$ is $-\infty$ for any q. In this case we prove the existence of traveling waves by minimizing the functional $I(\psi) = -Q(\psi) + \int_{\mathbf{R}^N} V(|\psi|^2)\,dx$ (or, equivalently, the functional $E - Q$) under the constraint $\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx = k$. More precisely, we have the following results:

Theorem 1.2 *Assume that $N \ge 2$ and (A1), (A2) are satisfied. For $k \ge 0$, let*

$$I_{min}(k) = \inf\Big\{I(\psi) \mid \psi \in \mathcal{E},\ \int_{\mathbf{R}^N} |\nabla\psi|^2\,dx = k\Big\}.$$

Then there is $k_\infty \in (0, \infty]$ such that the following holds:

(i) For any $k > k_\infty$, $I_{min}(k) = -\infty$. The function I_{min} is concave, decreasing on $[0, k_\infty)$, $I_{min}(k) \le -k/v_s^2$ for any $k \ge 0$, the right derivative of I_{min} at 0 is $-1/v_s^2$, and $\frac{I_{min}(k)}{k} \longrightarrow -\infty$ as $k \longrightarrow \infty$.

(ii) Let $k_0 = \inf\{k > 0 \mid I_{min}(k) < -k/v_s^2\} \in [0, k_\infty]$. For any $k \in (k_0, k_\infty)$, all sequences $(\psi_n)_{n\ge1} \subset \mathcal{E}$ satisfying $\int_{\mathbf{R}^N} |\nabla\psi_n|^2\,dx \longrightarrow k$ and $I(\psi_n) \longrightarrow I_{min}(k)$ are precompact for d_0 (modulo translations). If $\psi_k \in \mathcal{E}$ is a minimizer for $I_{min}(k)$, there exists $c = c(\psi_k) \in \left[\sqrt{-1/d^+I_{min}(k)}, \sqrt{-1/d^-I_{min}(k)}\right]$ such that $\psi_k(\frac{\cdot}{c})$ is a non constant traveling wave of (1.1) of speed $c(\psi_k)$.

(iii) We have $k_\infty < \infty$ if and only if ($N = 2$ and $\inf V < 0$). If $k_\infty = \infty$, the speeds of the traveling waves obtained from minimizers of $I_{min}(k)$ tend to 0 as $k \longrightarrow \infty$.

(iv) For $N \geq 3$, we have $k_0 > 0$. If $N = 2$ and assumption (A4) is satisfied we have $k_0 = 0$ if and only if $F''(1) \neq 3$, in which case the speeds of the traveling waves obtained from minimizers of $I_{min}(k)$ tend to v_s as $k \longrightarrow 0$.

In space dimension two, the tools developed to prove Theorem 1.2 enable us to find minimizers of E at fixed momentum on a subset of $\mathcal{E}$ even if V achieves negative values. We have:

Theorem 1.3 *Assume that $N = 2$ and that (A1), (A2) are satisfied. Let*

$$E^\sharp_{min}(q) = \inf \Big\{ E(\psi) \mid \psi \in \mathcal{E},\ Q(\psi) = q \ \text{ and } \ \int_{\mathbf{R}^2} V(|\psi|^2)dx \geq 0 \Big\}.$$

Then:

(i) The function $E^\sharp_{min}$ is concave, nondecreasing on $[0, \infty)$, $E^\sharp_{min}(q) \leq v_s q$, $d^+ E^\sharp_{min}(0) = v_s$ and $E^\sharp_{min}(q) \leq k_\infty$ for any $q > 0$, where k_∞ is as in Theorem 1.2.

(ii) Let $q^\sharp_0 = \inf\{q > 0 \mid E^\sharp_{min}(q) < v_s q\} \in [0, \infty]$ and $q^\sharp_\infty = \sup\{q > 0 \mid E^\sharp_{min}(q) < k_\infty\} \in (0, \infty]$. Then $q^\sharp_0 \leq q^\sharp_\infty$ and for any $q \in (q^\sharp_0, q^\sharp_\infty)$, all sequences $(\psi_n)_{n\geq1} \subset \mathcal{E}$ satisfying $Q(\psi_n) \longrightarrow q$ and $E(\psi_n) \longrightarrow E^\sharp_{min}(q)$ are precompact for d_0 (modulo translations).

The set $\mathcal{S}^\sharp_q = \{\psi \in \mathcal{E} \mid Q(\psi) = q,\ E(\psi) = E^\sharp_{min}(q)\}$ is not empty and is orbitally stable by the flow of (1.1) for the semi-distance d_0.

(iii) Any $\psi_q \in \mathcal{S}^\sharp_q$ verifies $\int_{\mathbf{R}^2} V(|\psi_q|^2)dx > 0$, hence minimizes E under the constraint $Q = q$ in the open set $\{w \in \mathcal{E} \mid \int_{\mathbf{R}^2} V(|w|^2)dx > 0\}$. Therefore, it is a traveling wave for (1.1) of speed $c(\psi_q) \in [d^+ E^\sharp_{min}(q), d^- E^\sharp_{min}(q)]$.

(iv) If assumption (A4) is satisfied, we have $q^\sharp_0 = 0$ if and only if $F''(1) \neq 3$, and in this case $c(\psi_q) \longrightarrow v_s$ as $q \longrightarrow 0$.

We may observe that in Theorem 1.2 it may happen that $k_0 = k_\infty$, and then (ii) never occurs. Statements (iii) and (iv) in Theorem 1.2 provide sufficient conditions to have $k_0 < k_\infty$. Actually, this is always the case if $N \geq 3$. In the case $N = 2$, we have $k_0 < k_\infty$ if $\inf V \geq 0$, or if ($\inf V < 0$, F verifies assumption (A4) and $F''(1) \neq 3$). Notice that the main physical example of nonlinearity satisfying $\inf V < 0$ is the cubic-quintic nonlinearity, for which one has $F''(1) \neq 3$. In the same way, in Theorem 1.3 it may happen that $q^\sharp_0 = q^\sharp_\infty$, in which case (ii) never holds, but here again, under assumption (A4), this is possible only if $F''(1) = 3$.

We conclude with a result concerning the nonexistence of small energy solutions to (1.3). This is a sharp version of a result proven in [7] for the Gross-Pitaevskii nonlinearity in dimension $N = 3$, then extended to $N \geq 4$ in [21]. The cases where $q_0 > 0$, $k_0 > 0$ or $q^\sharp_0 > 0$ in the above theorems follow directly from this result.

Proposition 1.4 *Assume that $N \geq 2$ and that F verifies (A1) and ((A2) or (A3)). Suppose that either*

- *$N \geq 3$, or*
- *$N = 2$, F satisfies (A4) and $F''(1) = 3$.*

The following holds.

(i) There is $k_* > 0$, depending only on N and F, such that if $c \in [0, v_s]$ and if $U \in \mathcal{E}$ is a solution to (1.3) satisfying $\int_{\mathbf{R}^N} |\nabla U|^2\, dx < k_*$, then U is constant.

(ii) Assume, moreover, that F satisfies (A2) with $p_0 < \frac{2}{N}$ or F satisfies (A3). There is $\ell_* > 0$, depending only on N and F, such that any solution $U \in \mathcal{E}$ to (1.3) with $c \in [0, v_s]$ and $\int_{\mathbf{R}^N} \left(|U|^2 - 1\right)^2 dx < \ell_*$ is constant.

Outline of the paper. In the next section we give a convenient definition of the momentum and we study its basic properties. In section 3 we present a regularization procedure which enables us to eliminate the small-scale topological defects of functions in $\mathcal{E}$. The tools introduced in sections 2 and 3 will be crucial for the variational machinery developed later. In section 4 we consider the problem of minimizing the energy at fixed momentum and we prove Theorem 1.1. We also develop some analytical tools that will be useful elsewhere. In section 5 we consider the problem of minimizing the functional $E - Q$ when the kinetic energy is fixed and we prove Theorem 1.2. Section 6 is devoted to the proof of Theorem 1.3. The orbital stability of the set of traveling waves provided by Theorems 1.1 and 1.3 is proven in section 7. In section 8 we investigate the relationship between the traveling waves given by Theorems 1.1, 1.2, 1.3 above and those found in [43]. In section 9 we show that if the stationary variant of (1.1) admits nontrivial solutions, the traveling waves found in the present paper converge to the ground states of the stationary equation as $c \longrightarrow 0$. The small energy solutions are studied in section 10, where we prove Proposition 1.4.

2 The momentum

The momentum (with respect to the x_1 direction) should be a functional defined on $\mathcal{E}$ whose "Gâteaux differential"[1] is $2i\partial_{x_1}$. In dimension $N \geq 3$, it has been shown in [43] how to define the momentum on $\mathcal{X}$ (and, consequently, on $\mathcal{E}$). In this section we will extend that definition in dimension $N = 2$.

It is clear that on the affine space $1 + H^1(\mathbf{R}^N) \subset \mathcal{E}$, the momentum should be defined by $Q(1+u) = \int_{\mathbf{R}^N} \langle iu_{x_1}, u\rangle \, dx$. In order to define the momentum on the whole $\mathcal{E}$, we introduce the space $\mathcal{Y} = \{\partial_{x_1}\phi \mid \phi \in \dot{H}^1(\mathbf{R}^N)\}$. It is easy to see that $\mathcal{Y}$ endowed with the norm $\|\partial_{x_1}\phi\|_{\mathcal{Y}} = \|\nabla\phi\|_{L^2(\mathbf{R}^N)}$ is a Hilbert space.

In dimension $N \geq 3$, it follows from Lemmas 2.1 and 2.2 in [43] that for any $u \in \mathcal{X}$ we have $\langle iu_{x_1}, u\rangle \in L^1(\mathbf{R}^N) + \mathcal{Y}$. If $N \geq 3$ and $\psi \in \mathcal{E}$, we have already seen there are $u \in \mathcal{X}$ and $\alpha_0 \in [0, 2\pi)$ such that $\psi = e^{i\alpha_0}(1+u)$. An easy computation gives $\langle i\psi_{x_1}, \psi\rangle = Im(u_{x_1}) + \langle iu_{x_1}, u\rangle$ and it is obvious that $Im(u_{x_1}) \in \mathcal{Y}$, thus $\langle i\psi_{x_1}, \psi\rangle \in L^1(\mathbf{R}^N) + \mathcal{Y}$. The next Lemma shows that a similar result holds if $N = 2$.

Lemma 2.1 *Let $N=2$. For any $\psi \in \mathcal{E}$ we have $|\psi|^2 - 1 \in L^2(\mathbf{R}^2)$ and $\langle i\psi_{x_1}, \psi\rangle \in L^1(\mathbf{R}^2) + \mathcal{Y}$.*

Proof. The following facts, borrowed from [12], will be useful here and in the sequel: for any $q \in [2, \infty)$ there is $C_q > 0$ such that for all $\phi \in L^1_{loc}(\mathbf{R}^2)$ satisfying $\nabla\phi \in L^2(\mathbf{R}^2)$ and $\mathcal{L}^2(supp(\phi)) < \infty$ we have

$$\|\phi\|_{L^q(\mathbf{R}^2)} \leq C_q \|\nabla\phi\|_{L^1(\mathbf{R}^2)}^{\frac{2}{q}} \|\nabla\phi\|_{L^2(\mathbf{R}^2)}^{1-\frac{2}{q}} \tag{2.1}$$

(see inequality (3.12) p. 108 in [12]). Since $\nabla\phi = 0$ a.e. on $\{\phi = 0\}$, (2.1) and the Cauchy-Schwarz inequality give

$$\|\phi\|_{L^q(\mathbf{R}^2)} \leq C_q \|\nabla\phi\|_{L^2(\mathbf{R}^2)} \left(\mathcal{L}^2(\{\phi(x) \neq 0\})\right)^{\frac{1}{q}}. \tag{2.2}$$

[1] We did not introduce a manifold structure on $\mathcal{E}$, although this can be done in a natural way, see [24, 25]. However, it will be clear (see (2.11)) what we mean here by "Gâteaux differential."

Notice that (2.2), which is a variant of inequality (3.10) p. 107 in [12], holds for any $q \in [1, \infty)$.

Let $\psi \in \mathcal{E}$. It is clear that

$$(2.3) \qquad \int_{\{|\psi| \le 2\}} (|\psi|^2 - 1)^2 \, dx = \int_{\{|\psi| \le 2\}} (\varphi^2(|\psi|) - 1)^2 \, dx < \infty.$$

Obviously, $\mathcal{L}^2(\{|\psi| \ge \frac{3}{2}\}) < \infty$ (because $E_{GL}(\psi) < \infty$) and $|\psi|^2 - 1 \le C(|\psi| - \frac{3}{2})^2$ on $\{|\psi| \ge 2\}$. Using (2.2) for $\phi = (|\psi| - \frac{3}{2})_+$ (which satisfies $|\nabla \phi| \le |\nabla \psi| 1\!\!1_{\{|\psi| \ge \frac{3}{2}\}}$ a.e.) we get

$$(2.4) \qquad \int_{\{|\psi| > 2\}} \left(|\psi|^2 - 1\right)^2 dx \le C \int \left(|\psi| - \frac{3}{2}\right)_+^4 dx \le C \|\nabla \psi\|^4_{L^2(\mathbf{R}^2)} \mathcal{L}^2(\{|\psi| \ge \frac{3}{2}\}) < \infty.$$

Thus $|\psi|^2 - 1 \in L^2(\mathbf{R}^2)$.

It follows from Theorem 1.8 p. 134 in [25] that there exist $w \in H^1(\mathbf{R}^2)$ and a real-valued function ϕ on $\mathbf{R}^2$ such that $\phi \in L^2_{loc}(\mathbf{R}^2)$, $\partial^\alpha \phi \in L^2(\mathbf{R}^2)$ for any $\alpha \in \mathbf{N}^2$ with $|\alpha| \ge 1$ and

$$(2.5) \qquad \psi = e^{i\phi} + w.$$

A simple computation gives

$$(2.6) \qquad \langle i\psi_{x_1}, \psi \rangle = -\frac{\partial \phi}{\partial x_1} + \frac{\partial}{\partial x_1}\left(\langle iw, e^{i\phi} \rangle \right) - 2\langle \frac{\partial \phi}{\partial x_1} e^{i\phi}, w \rangle + \langle i w_{x_1}, w \rangle.$$

The Cauchy-Schwarz inequality implies that $\langle \phi_{x_1} e^{i\phi}, w \rangle$ and $\langle i w_{x_1}, w \rangle$ belong to $L^1(\mathbf{R}^2)$. It is obvious that $\frac{\partial \phi}{\partial x_1} \in \mathcal{Y}$. We have $\langle iw, e^{i\phi} \rangle \in L^2(\mathbf{R}^2)$ and

$$\frac{\partial}{\partial x_j}\left(\langle iw, e^{i\phi} \rangle \right) = \langle i \frac{\partial w}{\partial x_j}, e^{i\phi} \rangle + \langle w, \frac{\partial \phi}{\partial x_j} e^{i\phi} \rangle.$$

The fact that w and $\frac{\partial \phi}{\partial x_j}$ belong to $H^1(\mathbf{R}^2)$ and the Sobolev embedding give $w, \ \frac{\partial \phi}{\partial x_j} \in L^p(\mathbf{R}^2)$ for any $p \in [2, \infty)$, hence $\langle w, \frac{\partial \phi}{\partial x_j} e^{i\phi} \rangle \in L^p(\mathbf{R}^2)$ for any $p \in [1, \infty)$. Since $\langle i \frac{\partial w}{\partial x_j}, e^{i\phi} \rangle \in L^2(\mathbf{R}^2)$, we get $\frac{\partial}{\partial x_j}\left(\langle iw, e^{i\phi} \rangle \right) \in L^2(\mathbf{R}^2)$, hence $\langle iw, e^{i\phi} \rangle \in H^1(\mathbf{R}^2)$ and consequently $\frac{\partial}{\partial x_1}\left(\langle iw, e^{i\phi} \rangle \right) \in \mathcal{Y}$. The proof of Lemma 2.1 is complete. □

For $v \in L^1(\mathbf{R}^N)$ and $w \in \mathcal{Y}$, let $L(v + w) = \int_{\mathbf{R}^N} v(x)\, dx$. It follows from Lemma 2.3 in [43] that L is well-defined and that it is a continuous linear functional on $L^1(\mathbf{R}^N) + \mathcal{Y}$. Taking into account Lemma 2.1 and the above considerations, for any $N \ge 2$ we give the following

Definition 2.2 *Given $\psi \in \mathcal{E}$, the momentum of ψ with respect to the x_1-direction is*

$$Q(\psi) = L(\langle i\psi_{x_1}, \psi \rangle).$$

Notice that the momentum (with respect to the x_1-direction) has been defined in [43] for functions $u \in \mathcal{X}$ by $\tilde{Q}(u) = L(\langle i \frac{\partial u}{\partial x_1}, u \rangle)$. If $\psi = e^{i\alpha_0}(1 + u)$, it is easy to see that $Q(\psi) = \tilde{Q}(u)$.

If $\psi \in \mathcal{E}$ is symmetric with respect to x_1 (in particular, if ψ is radial), then $Q(\psi) = Q(\psi(-x_1, x')) = -Q(\psi)$, hence $Q(\psi) = 0$.

If $\psi \in \mathcal{E}$ has a lifting $\psi = \rho e^{i\theta}$ with $\rho^2 - 1 \in L^2(\mathbf{R}^N)$ and $\theta \in \dot{H}^1(\mathbf{R}^N)$ (note that if $2 \le N \le 4$ we have always $|\psi|^2 - 1 \in L^2(\mathbf{R}^N)$ by (1.10) and the Sobolev embedding), then

$$(2.7) \qquad Q(\psi) = L(-\rho^2 \theta_{x_1}) = -\int_{\mathbf{R}^N} (\rho^2 - 1)\theta_{x_1}\, dx.$$

The next Lemma is an "integration by parts" formula.

Lemma 2.3 *For any* $\psi \in \mathcal{E}$ *and* $v \in H^1(\mathbf{R}^N)$ *we have* $\langle i\psi_{x_1}, v\rangle \in L^1(\mathbf{R}^N)$, $\langle i\psi, v_{x_1}\rangle \in L^1(\mathbf{R}^N) + \mathcal{Y}$ *and*

$$L(\langle i\psi_{x_1}, v\rangle + \langle i\psi, v_{x_1}\rangle) = 0. \tag{2.8}$$

Proof. If $N \geq 3$ this follows immediately from Lemma 2.5 in [43]. We give the proof in the case $N = 2$. The Cauchy-Schwarz inequality implies $\langle i\psi_{x_1}, v\rangle \in L^1(\mathbf{R}^2)$. Let $w \in H^1(\mathbf{R}^N)$ and ϕ be as in (2.5), so that $\psi = e^{i\phi} + w$. Then

$$\langle i\psi, v_{x_1}\rangle = \frac{\partial}{\partial x_1}\Big(\langle ie^{i\phi}, v\rangle\Big) + \langle \phi_{x_1} e^{i\phi}, v\rangle + \langle iw, v_{x_1}\rangle. \tag{2.9}$$

From the Cauchy-Schwarz inequality we have $\langle \phi_{x_1} e^{i\phi}, v\rangle \in L^1(\mathbf{R}^2)$ and $\langle iw, v_{x_1}\rangle \in L^1(\mathbf{R}^2)$. As in the proof of Lemma 2.1 we obtain $\langle ie^{i\phi}, v\rangle \in H^1(\mathbf{R}^2)$, hence $\frac{\partial}{\partial x_1}\left(\langle ie^{i\phi}, v\rangle\right) \in \mathcal{Y}$. We conclude that $\langle i\psi, v_{x_1}\rangle \in L^1(\mathbf{R}^N) + \mathcal{Y}$. Using (2.5), (2.9) and the definition of L we get

$$L(\langle i\psi_{x_1}, v\rangle + \langle i\psi, v_{x_1}\rangle) = L(\langle iw_{x_1}, v\rangle + \langle iw, v_{x_1}\rangle) = \int_{\mathbf{R}^N} \langle iw_{x_1}, v\rangle + \langle iw, v_{x_1}\rangle \, dx$$

and the last quantity is zero by the standard integration by parts formula for functions in $H^1(\mathbf{R}^2)$ (see, e.g., [10] p. 197). $\square$

Corollary 2.4 *Let* $\psi_1, \ \psi_2 \in \mathcal{E}$ *be such that* $\psi_1 - \psi_2 \in L^2(\mathbf{R}^N)$*. Then*

$$|Q(\psi_1) - Q(\psi_2)| \leq \|\psi_1 - \psi_2\|_{L^2(\mathbf{R}^N)} \left(\left\|\frac{\partial \psi_1}{\partial x_1}\right\|_{L^2(\mathbf{R}^N)} + \left\|\frac{\partial \psi_2}{\partial x_1}\right\|_{L^2(\mathbf{R}^N)} \right) \tag{2.10}$$

Proof. The same as the proof of Corollary 2.6 in [43]. $\square$

Let $\psi \in \mathcal{E}$. It is easy to see that for any function with compact support $\phi \in H^1(\mathbf{R}^N)$ we have $\psi + \phi \in \mathcal{E}$ and using Lemma 2.3 we get

$$\lim_{t\to 0} \frac{1}{t}(Q(\psi + t\phi) - Q(\psi)) = L(\langle i\psi_{x_1}, \phi\rangle + \langle i\phi_{x_1}, \psi\rangle) = 2\int_{\mathbf{R}^N} \langle i\psi_{x_1}, \phi\rangle \, dx. \tag{2.11}$$

The momentum has a nice behavior with respect to dilations: for $\psi \in \mathcal{E}$, $\lambda, \ \sigma > 0$ we have

$$Q(\psi_{\lambda,\sigma}) = \sigma^{N-1} Q(\psi). \tag{2.12}$$

3 A regularization procedure

The regularization procedure described below will be an important tool for our analysis. It was first introduced in [2], then developed in [43], where it was a key ingredient in proofs. It enables us to get rid of the small-scale topological defects of functions and in the meantime to control the Ginzburg-Landau energy and the momentum of the regularized functions.

In this section Ω is an open set in $\mathbf{R}^N$. We do not assume Ω bounded, nor connected. If $\partial\Omega \neq \emptyset$, we assume that $\partial\Omega$ is C^2. Fix $\psi \in \mathcal{E}$ and $h > 0$. We consider the functional

$$G^{\psi}_{h,\Omega}(\zeta) = \begin{cases} E^{\Omega}_{GL}(\zeta) + \dfrac{1}{h^2}\displaystyle\int_\Omega |\zeta - \psi|^2 \, dx & \text{if } N = 2, \\[2ex] E^{\Omega}_{GL}(\zeta) + \dfrac{1}{h^2}\displaystyle\int_\Omega \varphi\left(|\zeta - \psi|^2\right) dx & \text{if } N \geq 3. \end{cases}$$

Note that $G^{\psi}_{h,\Omega}(\zeta)$ may equal ∞ for some $\zeta \in \mathcal{E}$; however, $G^{\psi}_{h,\Omega}(\zeta)$ is finite whenever $\zeta \in \mathcal{E}$ and $\zeta - \psi \in L^2(\Omega)$. We denote $H^1_0(\Omega) = \{u \in H^1(\mathbf{R}^N) \mid u = 0 \text{ on } \mathbf{R}^N \setminus \Omega\}$ and

$$H^1_{\psi}(\Omega) = \{\zeta \in \mathcal{E} \mid \zeta - \psi \in H^1_0(\Omega)\}.$$

Assume that $N \geq 3$ and $\psi = e^{i\alpha_0}(1+u) \in \mathcal{E}$, where $\alpha_0 \in [0, 2\pi)$ and $u \in \mathcal{X}$. Then

$$H^1_{\psi}(\Omega) = \{e^{i\alpha_0}(1+v) \mid v \in H^1_u(\Omega)\}.$$

Let

$$\tilde{G}^u_{h,\Omega}(w) = E^{\Omega}_{GL}(1+w) + \frac{1}{h^2}\int_{\Omega} \varphi\left(|w-u|^2\right)\, dx.$$

It is obvious that $\zeta = e^{i\alpha_0}(1+v)$ is a minimizer of $G^{\psi}_{h,\Omega}$ in $H^1_{\psi}(\Omega)$ if and only if v is a minimizer of $\tilde{G}^u_{h,\Omega}$ in $H^1_u(\Omega)$, hence the results proved in [43] for minimizers of $\tilde{G}^u_{h,\Omega}$ also hold for minimizers of $G^{\psi}_{h,\Omega}$.

The next three lemmas are analogous to Lemmas 3.1, 3.2 and 3.3 in [43]. For the convenience of the reader we give the full statements in any space dimension, but for the proofs in the case $N \geq 3$ we refer to [43]; we only indicate here what changes in proofs if $N = 2$.

Lemma 3.1 *(i) The functional $G^{\psi}_{h,\Omega}$ has a minimizer in $H^1_{\psi}(\Omega)$.*

(ii) Let ζ_h be a minimizer of $G^{\psi}_{h,\Omega}$ in $H^1_{\psi}(\Omega)$. There exist constants $C_i > 0$, depending only on N, such that:

$$E^{\Omega}_{GL}(\zeta_h) \leq E^{\Omega}_{GL}(\psi); \tag{3.1}$$

$$\|\zeta_h - \psi\|^2_{L^2(\Omega)} \leq \begin{cases} h^2 E^{\Omega}_{GL}(\psi) & \text{if } N = 2, \\ h^2 E^{\Omega}_{GL}(\psi) + C_1\left(E^{\Omega}_{GL}(\psi)\right)^{1+\frac{2}{N}} h^{\frac{4}{N}} & \text{if } N \geq 3. \end{cases} \tag{3.2}$$

$$\int_{\Omega} \left| \left(\varphi^2(|\zeta_h|) - 1\right)^2 - \left(\varphi^2(|\psi|) - 1\right)^2 \right| dx \leq C_2 h E^{\Omega}_{GL}(u); \tag{3.3}$$

$$|Q(\zeta_h) - Q(\psi)| \leq \begin{cases} 2h E^{\Omega}_{GL}(\psi) & \text{if } N = 2, \\ C_3\left(h^2 + \left(E^{\Omega}_{GL}(\psi)\right)^{\frac{2}{N}} h^{\frac{4}{N}}\right)^{\frac{1}{2}} E^{\Omega}_{GL}(\psi) & \text{if } N \geq 3. \end{cases} \tag{3.4}$$

(iii) For $z \in \mathbf{C}$, denote $H(z) = \left(\varphi^2(|z|) - 1\right)\varphi(|z|)\varphi'(|z|)\frac{z}{|z|}$ if $z \neq 0$ and $H(0) = 0$. Then any minimizer ζ_h of $G^{\psi}_{h,\Omega}$ in $H^1_{\psi}(\Omega)$ satisfies in $\mathcal{D}'(\Omega)$ the equation

$$\begin{cases} -\Delta\zeta_h + H(\zeta_h) + \dfrac{1}{h^2}(\zeta_h - \psi) = 0 & \text{if } N = 2, \\ -\Delta\zeta_h + H(\zeta_h) + \dfrac{1}{h^2}\varphi'\left(|\zeta_h - \psi|^2\right)(\zeta_h - \psi) = 0 & \text{if } N \geq 3. \end{cases} \tag{3.5}$$

Moreover, for any $\omega \subset\subset \Omega$ we have $\zeta_h \in W^{2,p}(\omega)$ for $p \in [1, \infty)$; thus, in particular, $\zeta_h \in C^{1,\alpha}(\omega)$ for $\alpha \in [0, 1)$.

(iv) For any $h > 0$, $\delta > 0$ and $R > 0$ there exists a constant $K = K(N, h, \delta, R) > 0$ such that for any $\psi \in \mathcal{E}$ with $E^{\Omega}_{GL}(\psi) \leq K$ and for any minimizer ζ_h of $G^{\psi}_{h,\Omega}$ in $H^1_{\psi}(\Omega)$ we have

$$1 - \delta < |\zeta_h(x)| < 1 + \delta \qquad \text{whenever } x \in \Omega \text{ and } dist(x, \partial\Omega) > 4R. \tag{3.6}$$

Proof. Let $N = 2$.

(i) The existence of a minimizer is proven exactly as in Lemma 3.1 in [43].

(ii) Let ζ_h be a minimizer. We have $G^{\psi}_{h,\Omega}(\zeta_h) \leq G^{\psi}_{h,\Omega}(\psi) = E_{GL}(\psi)$ and this gives (3.1) and (3.2). It is obvious that

$$\left| \left(\varphi^2(|z_1|) - 1\right)^2 - \left(\varphi^2(|z_2|) - 1\right)^2 \right| \leq 6\big|\varphi(|z_1|) - \varphi(|z_2|)\big| \cdot \big|\varphi(|z_1|^2) + \varphi(|z_2|^2) - 2\big|$$

and $|\varphi(|z_1|) - \varphi(|z_2|)| \leq |z_1 - z_2|$. Using the Cauchy-Schwarz inequality and (3.2) we get

$$\int_{\Omega} \left| \left(\varphi^2(|\zeta_h|) - 1\right)^2 - \left(\varphi^2(|\psi|) - 1\right)^2 \right| dx$$

$$\leq 6\|\zeta_h - \psi\|_{L^2(\Omega)} \left(\int_{\Omega} \left|\varphi^2(|\zeta_h|) + \varphi^2(|\psi|) - 2\right|^2 dx \right)^{\frac{1}{2}}$$

$$\leq 6h \left(E^{\Omega}_{GL}(\psi)\right)^{\frac{1}{2}} \cdot \left(2\int_{\Omega} \left(\varphi^2(|\zeta_h|) - 1\right)^2 + \left(\varphi^2(|\psi|) - 1\right)^2 \, dx \right)^{\frac{1}{2}} \leq 12\sqrt{2}hE^{\Omega}_{GL}(\psi)$$

and (3.3) is proven. Finally, (3.4) follows from Corollary 2.4, (3.1) and (3.2).

(iii) For any $\phi \in C^{\infty}_c(\Omega)$ we have $\zeta_h + \phi \in H^1_{\psi}(\Omega)$ and the function $t \longmapsto G^{\psi}_{h,\Omega}(\zeta_h + t\phi)$ is differentiable and achieves its minimum at $t = 0$. Hence $\frac{d}{dt}\Big|_{t=0} \left(G^{\psi}_{h,\Omega}(\zeta_h + t\phi)\right) = 0$ for any $\phi \in C^{\infty}_c(\Omega)$ and this is precisely (3.5).

For any $z \in \mathbf{C}$ we have

$$|H(z)| \leq 3|\varphi^2(|z|) - 1| \leq 24. \tag{3.7}$$

Since $\zeta_h \in \mathcal{E}$, we have $\varphi^2(|\zeta_h|) - 1 \in L^2(\mathbf{R}^2)$ and the previous inequality gives $H(v_h) \in L^2 \cap L^{\infty}(\mathbf{R}^2)$. We have $\zeta_h, \psi \in H^1_{loc}(\mathbf{R}^2)$ and from the Sobolev embedding theorem we get $\zeta_h, \psi \in L^p_{loc}(\mathbf{R}^2)$ for any $p \in [2, \infty)$. Using (3.5) we infer that $\Delta\zeta_h \in L^p_{loc}(\Omega)$ for any $p \in [2, \infty)$. Then (iii) follows from standard elliptic estimates (see, e.g., Theorem 9.11 p. 235 in [26]).

iv) Using (3.7) we get

$$\|H(\zeta_h)\|_{L^2(\Omega)} \leq 3\|\varphi^2(|\zeta_h|) - 1\|_{L^2(\Omega)} \leq 3\sqrt{2}\left(E^{\Omega}_{GL}(\zeta_h)\right)^{\frac{1}{2}} \leq 3\sqrt{2}\left(E^{\Omega}_{GL}(\psi)\right)^{\frac{1}{2}}.$$

From (3.5), (3.2) and the above estimate we get

$$\|\Delta\zeta_h\|_{L^2(\Omega)} \leq \left(3\sqrt{2} + \frac{1}{h}\right)\left(E^{\Omega}_{GL}(\psi)\right)^{\frac{1}{2}}. \tag{3.8}$$

For a measurable set $\omega \subset \mathbf{R}^N$ with $\mathcal{L}^N(\omega) < \infty$ and for $f \in L^1(\omega)$, we denote by $m(f, \omega) = \frac{1}{\mathcal{L}^N(\omega)} \int_{\omega} f(x)\, dx$ the mean value of f on ω. In particular, if $f \in L^2(\omega)$ using the Cauchy-Schwarz inequality we get $|m(f, \omega)| \leq \left(\mathcal{L}^N(\omega)\right)^{-\frac{1}{2}} \|f\|_{L^2(\omega)}$ and consequently

$$\|m(f, \omega)\|_{L^q(\omega)} = \left(\mathcal{L}^N(\omega)\right)^{\frac{1}{q}} |m(f, \omega)| \leq \left(\mathcal{L}^N(\omega)\right)^{\frac{1}{q} - \frac{1}{2}} \|f\|_{L^2(\omega)}. \tag{3.9}$$

Let x_0 be such that $B(x_0, 4R) \subset \Omega$. Using the Poincaré inequality and (3.1) we have

$$\|\zeta_h - m(\zeta_h, B(x_0, 4R))\|_{L^2(B(x_0,4R))} \leq C_P R\|\nabla\zeta_h\|_{L^2(B(x_0,4R))} \leq C_P R\left(E^{\Omega}_{GL}(\psi)\right)^{\frac{1}{2}}. \tag{3.10}$$

It is well-known (see Theorem 9.11 p. 235 in [26]) that for $p \in (1,\infty)$ there exists $C = C(N,r,p) > 0$ such that for any $w \in W^{2,p}(B(a,2r))$ we have

$$\|w\|_{W^{2,p}(B(a,r))} \leq C \left(\|w\|_{L^p(B(a,2r))} + \|\Delta w\|_{L^p(B(a,2r))} \right). \tag{3.11}$$

From (3.8), (3.10) and (3.11) we get

$$\|\zeta_h - m(\zeta_h, B(x_0,4R))\|_{W^{2,2}(B(x_0,2R))} \leq C(h,R) \left(E^{\Omega}_{GL}(\psi) \right)^{\frac{1}{2}} \tag{3.12}$$

and in particular

$$\forall 1 \leq i,j \leq 2, \qquad \left\| \frac{\partial^2 \zeta_h}{\partial x_i \partial x_j} \right\|_{L^2(B(x_0,2R))} \leq C(h,R) \left(E^{\Omega}_{GL}(\psi) \right)^{\frac{1}{2}}. \tag{3.13}$$

We will use the following variant of the Gagliardo-Nirenberg inequality:

$$\|w - m(w, B(a,r))\|_{L^p(B(a,r))} \leq C(p,q,N,r) \|w\|^{\frac{q}{p}}_{L^q(B(a,2r))} \|\nabla w\|^{1-\frac{q}{p}}_{L^N(B(a,2r))} \tag{3.14}$$

for any $w \in W^{1,N}(B(a,2r))$, where $1 \leq q \leq p < \infty$ (see, e.g., [33] p. 78). Using (3.14) with $N=2$, $p=4$, $q=2$, then (3.1) and (3.13) we find

$$\begin{aligned} \|\nabla \zeta_h - m(\nabla \zeta_h, B(x_0,R))\|_{L^4(B(x_0,R))} &\leq C \|\nabla \zeta_h\|^{\frac{1}{2}}_{L^2(B(x_0,2R))} \|\nabla^2 \zeta_h\|^{\frac{1}{2}}_{L^2(B(x_0,2R))} \\ &\leq C(h,R) \left(E^{\Omega}_{GL}(\psi) \right)^{\frac{1}{2}}. \end{aligned} \tag{3.15}$$

By (3.9) and (3.1) we have $\|m(\nabla \zeta_h, B(x_0,R))\|_{L^4(B(x_0,R))} \leq (\pi R^2)^{-\frac{1}{4}} \left(E^{\Omega}_{GL}(\psi) \right)^{\frac{1}{2}}$. Together with (3.15), this gives

$$\|\nabla \zeta_h\|_{L^4(B(x_0,R))} \leq C(h,R) \left(E^{\Omega}_{GL}(\psi) \right)^{\frac{1}{2}}. \tag{3.16}$$

We will use the Morrey inequality which asserts that, for any $w \in C^0 \cap W^{1,p}(B(x_0,r))$ with $p > N$ we have

$$|w(x) - w(y)| \leq C(p,N) |x-y|^{1-\frac{N}{p}} ||\nabla w||_{L^p(B(x_0,r))} \qquad \text{for any } x,y \in B(x_0,r) \tag{3.17}$$

(see the proof of Theorem IX.12 p. 166 in [10]). The Morrey inequality and (3.16) imply that

$$|\zeta_h(x) - \zeta_h(y)| \leq C_*(h,R) \left(E^{\Omega}_{GL}(\psi) \right)^{\frac{1}{2}} |x-y|^{\frac{1}{2}} \qquad \text{for any } x,y \in B(x_0,R). \tag{3.18}$$

Fix $\delta > 0$. Assume that there exists $x_0 \in \Omega$ such that $dist(x_0, \partial\Omega) > 4R$ and $\big| |\zeta_h(x_0)| - 1 \big| \geq \delta$. Since $\Big| \big| |\zeta_h(x)| - 1 \big| - \big| |\zeta_h(y)| - 1 \big| \Big| \leq |\zeta_h(x) - \zeta_h(y)|$, using (3.18) we infer that

$$\big| |\zeta_h(x)| - 1 \big| \geq \frac{\delta}{2} \qquad \text{for any } x \in B(x_0, r_\delta),$$

where $r_\delta = \min \left(R, \frac{\delta^2}{4C_*^2(h,R) E^{\Omega}_{GL}(\psi)} \right)$. Let

$$\eta(s) = \inf\{ (\varphi^2(\tau) - 1)^2 \mid \tau \in (-\infty, 1-s] \cup [1+s, \infty) \}. \tag{3.19}$$

It is clear that η is nondecreasing and positive on $(0,\infty)$. We have:

$$\begin{aligned} E^{\Omega}_{GL}(\psi) \geq E^{\Omega}_{GL}(\zeta_h) &\geq \tfrac{1}{2} \int_{B(x_0,r_\delta)} \left(\varphi^2(|\zeta_h|) - 1 \right)^2 dx \\ &\geq \tfrac{1}{2} \int_{B(x_0,r_\delta)} \eta(\tfrac{\delta}{2})\, dx = \tfrac{\pi}{2} \eta(\tfrac{\delta}{2}) r_\delta^2 = \tfrac{\pi}{2} \eta(\tfrac{\delta}{2}) \min \left(R, \tfrac{\delta^2}{4C_*^2(h,R) E^{\Omega}_{GL}(\psi)} \right)^2. \end{aligned} \tag{3.20}$$

It is clear that there exists a constant $K = K(h,R,\delta)$ such that (3.20) cannot hold if $E^{\Omega}_{GL}(\psi) \leq K$. We infer that $\big| |\zeta_h(x_0)| - 1 \big| < \delta$ whenever $x_0 \in \Omega$, $dist(x_0, \partial\Omega) > 4R$ and $E^{\Omega}_{GL}(\psi) \leq K$. $\square$

Lemma 3.2 *Let $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ be a sequence of functions satisfying:*

(a) $(E_{GL}(\psi_n))_{n\geq 1}$ is bounded and

(b) $\lim\limits_{n\to\infty} \Big(\sup\limits_{y\in\mathbf{R}^N} E_{GL}^{B(y,1)}(\psi_n) \Big) = 0.$

There exists a sequence $h_n \longrightarrow 0$ such that for any minimizer ζ_n of $G^{\psi_n}_{h_n,\mathbf{R}^N}$ in $H^1_{\psi_n}(\mathbf{R}^N)$ we have $\| |\zeta_n| - 1 \|_{L^\infty(\mathbf{R}^N)} \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. Let $N = 2$. We split the proof into several steps.

Step 1. Choice of the sequence $(h_n)_{n\geq 1}$. Let $M = \sup\limits_{n\geq 1} E_{GL}(\psi_n)$. For $n \geq 1$ and $x \in \mathbf{R}^2$ we denote

$$m_n(x) = m(\psi_n, B(x,1)) = \frac{1}{\pi}\int_{B(x,1)} \psi_n(y)\,dy.$$

The Poincaré inequality implies that there exists $C_P > 0$ such that

$$\int_{B(x,1)} |\psi_n(y) - m_n(x)|^2\,dy \leq C_P \int_{B(x,1)} |\nabla\psi_n|^2\,dy.$$

Using assumption (b) we find

$$\sup_{x\in\mathbf{R}^2} \|\psi_n - m_n(x)\|_{L^2(B(x,1))} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{3.21}$$

Proceeding exactly as in the proof of Lemma 3.2 in [43] (see the proof of (3.35) there) we get

$$\lim_{n\to\infty} \sup_{x\in\mathbf{R}^2} |H(m_n(x))| = 0. \tag{3.22}$$

Let

$$h_n = \max\left(\left(\sup_{x\in\mathbf{R}^2} \|\psi_n - m_n(x)\|_{L^2(B(x,1))} \right)^{\frac{1}{3}}, \sup_{x\in\mathbf{R}^2} |H(m_n(x))| \right). \tag{3.23}$$

From (3.21) and (3.22) it follows that $h_n \longrightarrow 0$ as $n \longrightarrow \infty$. Hence we may assume that $0 < h_n < 1$ for each n (if $h_n = 0$ then ψ_n is constant a.e. and any minimizer ζ_n of $G^{\psi_n}_{h_n,\mathbf{R}^2}$ equals ψ_n a.e.).

Let ζ_n be a minimizer of $G^{\psi_n}_{h_n,\mathbf{R}^2}$ (as given by Lemma 3.1 (i)). It follows from Lemma 3.1 (iii) that ζ_n satisfies (3.5) and $\zeta_n \in W^{2,2}_{loc}(\mathbf{R}^2)$.

Step 2. We prove that $\|\Delta\zeta_n\|_{L^2(B(x,\frac{1}{2}))}$ is bounded independently on n and on x. There is no loss of generality to assume that $x = 0$. Then we observe that (3.5) can be written as

$$-\Delta\zeta_n + \frac{1}{h_n^2}(\zeta_n - m_n(0)) = f_n \qquad \text{in } \mathcal{D}'(\mathbf{R}^2), \tag{3.24}$$

where

$$f_n = \frac{1}{h_n^2}(\psi_n - m_n(0)) - (H(\zeta_n) - H(m_n(0))) - H(m_n(0)). \tag{3.25}$$

From (3.2) we have $\|\zeta_n - \psi_n\|_{L^2(\mathbf{R}^2)} \leq h_n E_{GL}(\psi_n)^{\frac{1}{2}} \leq h_n M^{\frac{1}{2}}$ and from (3.23) we obtain $\|\psi_n - m_n(0)\|_{L^2(B(0,1))} \leq h_n^3 \leq h_n$, hence

$$\|\zeta_n - m_n(0)\|_{L^2(B(0,1))} \leq (M^{\frac{1}{2}} + 1)h_n. \tag{3.26}$$

Since H is Lipschitz, we get

$$\|H(\zeta_n) - H(m_n(0))\|_{L^2(B(0,1)} \le C_1 \|\zeta_n - m_n(0)\|_{L^2(B(0,1))} \le C_2 h_n. \tag{3.27}$$

Using (3.25), (3.23) and (3.27) we get

$$\begin{aligned} &\|f_n\|_{L^2(B(0,1))} \\ &\le \frac{1}{h_n^2}\|\psi_n - m_n(0)\|_{L^2(B(0,1))} + \|H(\zeta_n) - H(m_n(0))\|_{L^2(B(0,1))} + \pi^{\frac{1}{2}}|H(m_n(0))| \\ &\le C_3 h_n. \end{aligned} \tag{3.28}$$

It is obvious that for any bounded domain $\Omega \subset \mathbf{R}^2$, each term in (3.24) belongs to $H^{-1}(\Omega)$. Let $\chi \in C_c^\infty(\mathbf{R}^2)$ be such that $supp(\chi) \subset B(0,1)$, $0 \le \chi \le 1$ and $\chi = 1$ on $B(0, \frac{1}{2})$. Taking the duality product of (3.24) by $\chi(\zeta_n - m_n(0))$ we find

(3.29)

$$\int_{\mathbf{R}^2} \chi|\nabla\zeta_n|^2\,dx - \frac{1}{2}\int_{\mathbf{R}^2}(\Delta\chi)|\zeta_n - m_n(0)|^2\,dx + \frac{1}{h_n^2}\int_{\mathbf{R}^2}\chi|\zeta_n - m_n(0)|^2\,dx = \int_{\mathbf{R}^2}\langle f_n, \zeta_n - m_n(0)\rangle\chi\,dx.$$

Using (3.29), the Cauchy-Schwarz inequality and (3.26), (3.28) we infer that

(3.30)

$$\begin{aligned} &\frac{1}{h_n^2}\int_{B(0,\frac{1}{2})}|\zeta_n - m_n(0)|^2\,dx \\ &\le \|\Delta\chi\|_{L^\infty(\mathbf{R}^2)}\int_{B(0,1)}|\zeta_n - m_n(0)|^2\,dx + \|f_n\|_{L^2(B(0,1))}\|\zeta_n - m_n(0)\|_{L^2(B(0,1))} \le C_4 h_n^2. \end{aligned}$$

Now (3.24), (3.28) and (3.30) imply that there is $C_5 > 0$ such that $\|\Delta\zeta_n\|_{L^2(B(0,\frac{1}{2}))} \le C_5$. Thus we have proved that for any n and x,

$$\|\Delta\zeta_n\|_{L^2(B(x,\frac{1}{2}))} \le C_5, \qquad \text{where } C_5 \text{ does not depend on } x \text{ and } n. \tag{3.31}$$

Step 3. A Hölder estimate on ζ_n. It follows from (3.11) that

$$\|\zeta_n - m_n\|_{W^{2,2}(B(x,\frac{1}{4}))} \le C(\|\Delta\zeta_n\|_{L^2(B(x,\frac{1}{2}))} + \|\zeta_n - m_n\|_{L^2(B(x,\frac{1}{2}))}) \le C_6. \tag{3.32}$$

From (3.14) and (3.32) we find

$$\|\nabla\zeta_n - m(\nabla\zeta_n, B(x, \frac{1}{8}))\|_{L^4(B(x,\frac{1}{8}))} \le C\|\nabla\zeta_n\|_{L^2(B(x,\frac{1}{4}))}^{\frac{1}{2}}\|\nabla^2\zeta_n\|_{L^2(B(x,\frac{1}{4}))}^{\frac{1}{2}} \le C_7. \tag{3.33}$$

It is clear that $|m(\nabla\zeta_n, B(x, \frac{1}{8}))| \le \left(\mathcal{L}^2(B(x, \frac{1}{8}))\right)^{-\frac{1}{2}}\|\nabla\zeta_n\|_{L^2(B(x,\frac{1}{8}))} \le C_8$. Then (3.33) implies that $\|\nabla\zeta_n\|_{L^4(B(x,\frac{1}{8}))}$ is bounded independently on n and x. Using the Morrey inequality (3.17) we infer that there is $C_9 > 0$ such that

$$|\zeta_n(x) - \zeta_n(y)| \le C_9|x-y|^{\frac{1}{2}} \qquad \text{for any } n \in \mathbf{N}^* \text{ and any } x, y \in \mathbf{R}^2 \text{ with } |x-y| < \frac{1}{8}. \tag{3.34}$$

Step 4. Conclusion. Let $\delta_n = \|\,|\zeta_n| - 1\|_{L^\infty(\mathbf{R}^2)}$ if ζ_n is bounded, and $\delta_n = 1$ otherwise. Choose $x_0^n \in \mathbf{R}^2$ such that $\left|\,|\zeta_n(x_0^n)| - 1\right| \ge \frac{\delta_n}{2}$. From (3.34) we infer that $\left|\,|\zeta_n(x)| - 1\right| \ge \frac{\delta_n}{4}$ for any $x \in B(x_0^n, r_n)$, where $r_n = \min\left(\frac{1}{8}, \left(\frac{\delta_n}{4C_9}\right)^2\right)$. Let η be as in (3.19). Then we have

$$\int_{B(x_0^n,r_n)}\left(\varphi^2(|\zeta_n|) - 1\right)^2\,dx \ge \int_{B(x_0^n,r_n)}\eta\left(\frac{\delta_n}{4}\right)\,dx = \eta\left(\frac{\delta_n}{4}\right)\pi r_n^2. \tag{3.35}$$

On the other hand, the function $z \longmapsto \left(\varphi^2(|z|)-1\right)^2$ is Lipschitz on $\mathbf{C}$. From this fact, the Cauchy-Schwarz inequality, (3.2) and assumption (a) we get

$$\int_{B(x,1)} \Big| \left(\varphi^2(|\zeta_n(y)|)-1\right)^2 - \left(\varphi^2(|\psi_n(y)|)-1\right)^2 \Big| \, dy$$
$$\leq C \int_{B(x,1)} |\zeta_n(y)-\psi_n(y)| \, dy \leq C\pi^{\frac{1}{2}} \|\zeta_n-\psi_n\|_{L^2(B(x,1))} \leq C\pi^{\frac{1}{2}} \|\zeta_n-\psi_n\|_{L^2(\mathbf{R}^2)} \leq C_{10} h_n.$$

Then using assumption (b) we infer that

$$\sup_{x\in\mathbf{R}^2} \int_{B(x,1)} \left(\varphi^2(|\zeta_n(y)|)-1\right)^2 \, dy \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{3.36}$$

From (3.35) and (3.36) we get $\lim\limits_{n\to\infty} \eta\left(\frac{\delta_n}{4}\right) r_n^2 = 0$ and this clearly implies $\lim\limits_{n\to\infty} \delta_n = 0$. This completes the proof of Lemma 3.2. $\square$

The next result is based on Lemma 3.1 and will be very useful in the next sections to prove the "concentration" of minimizing sequences. For $0 < R_1 < R_2$ we denote $\Omega_{R_1,R_2} = B(0,R_2) \setminus \overline{B}(0,R_1)$.

Lemma 3.3 *Let $A > A_3 > A_2 > 1$. There exist $\varepsilon_0 > 0$ and $C_i > 0$, depending only on N, A, A_2, A_3 (and F for (vi)) such that for any $R \geq 1$, $\varepsilon \in (0,\varepsilon_0)$ and $\psi \in \mathcal{E}$ verifying $E_{GL}^{\Omega_{R,AR}}(\psi) \leq \varepsilon$, there exist two functions ψ_1, $\psi_2 \in \mathcal{E}$ and a constant $\theta_0 \in [0,2\pi)$ satisfying the following properties:*

(i) $\psi_1 = \psi$ on $B(0,R)$ and $\psi_1 = e^{i\theta_0}$ on $\mathbf{R}^N \setminus B(0,A_2R)$,

(ii) $\psi_2 = \psi$ on $\mathbf{R}^N \setminus B(0,AR)$ and $\psi_2 = e^{i\theta_0} =$ constant on $B(0,A_3R)$,

(iii) $\displaystyle\int_{\mathbf{R}^N} \left| \left|\frac{\partial\psi}{\partial x_j}\right|^2 - \left|\frac{\partial\psi_1}{\partial x_j}\right|^2 - \left|\frac{\partial\psi_2}{\partial x_j}\right|^2 \right| dx \leq C_1\varepsilon$ for $j = 1,\dots,N$,

(iv) $\displaystyle\int_{\mathbf{R}^N} \Big| \left(\varphi^2(|\psi|)-1\right)^2 - \left(\varphi^2(|\psi_1|)-1\right)^2 - \left(\varphi^2(|\psi_2|)-1\right)^2 \Big| \, dx \leq C_2\varepsilon$,

(v) $|Q(\psi)-Q(\psi_1)-Q(\psi_2)| \leq C_3\varepsilon$,

(vi) If assumptions (A1) and (A2) in the introduction hold, then

$$\int_{\mathbf{R}^N} \left| V(|\psi|^2) - V(|\psi_1|^2) - V(|\psi_2|^2) \right| dx \leq \begin{cases} C_4\varepsilon + C_5\sqrt{\varepsilon}\,(E_{GL}(\psi))^{\frac{2^*-1}{2}} & \text{if } N \geq 3, \\ C_6\varepsilon + C_7\sqrt{\varepsilon}\,(E_{GL}(\psi))^{p_0+1} & \text{if } N = 2. \end{cases}$$

Furthermore, the same estimate holds with V_+ (respectively V_-) instead of V.

Proof. If $N \geq 3$, this is Lemma 3.3 in [43].

Let $N = 2$. Fix $k > 0$, A_1 and A_4 such that $1 + 4k < A_1 < A_2 < A_3 < A_4 < A - 4k$. Let $h = 1$ and $\delta = \frac{1}{2}$. Let $K(N,h,\delta,r)$ be as in Lemma 3.1 (iv). We will prove that Lemma 3.3 holds for $\varepsilon_0 = \min\left(K(2,1,\frac{1}{2},k), \frac{\pi}{8}\ln\left(\frac{A-4k}{1+4k}\right)\right)$.

Fix $\varepsilon < \varepsilon_0$. Consider $\psi \in \mathcal{E}$ such that $E_{GL}^{\Omega_{R,AR}}(\psi) \leq \varepsilon$. Let ζ be a minimizer of $G^{\psi}_{1,\Omega_{R,AR}}$ in the space $H^1_{\psi}(\Omega_{R,AR})$. Such minimizers exist by Lemma 3.1 (but are perhaps not unique). From Lemma 3.1 (iii) we have $\zeta \in W^{2,p}_{loc}(\Omega_{R,AR})$ for any $p \in [1,\infty)$, hence $\zeta \in C^1(\Omega_{R,AR})$. Moreover, Lemma 3.1 (iv) implies that

$$\frac{1}{2} \leq |\zeta(x)| \leq \frac{3}{2} \qquad \text{for any } x \text{ such that } R + 4k \leq |x| \leq AR - 4k. \tag{3.37}$$

Therefore, the topological degree $deg(\frac{\zeta}{|\zeta|}, \partial B(0,r))$ is well defined for any $r \in [R+4k, AR-4k]$ and does not depend on r. It is well-known that ζ admits a C^1 lifting θ (i.e. $\zeta = |\zeta|e^{i\theta}$) on $\Omega_{R+4k,AR-4k}$ if and only if $deg(\zeta, \partial B(0,r)) = 0$ for $r \in (R+4k, AR-4k)$. Denoting by $\tau = (-\sin t, \cos t)$ the unit tangent vector at $\partial B(0,r)$ at a point $re^{it} = (r\cos t, r\sin t) \in \partial B(0,r)$, we get

$$
\begin{aligned}
|deg(\zeta, \partial B(0,r))| &= \left| \frac{1}{2i\pi} \int_0^{2\pi} \frac{\frac{\partial}{\partial t}(\zeta(re^{it}))}{\zeta(re^{it})} \, dt \right| = \left| \frac{r}{2i\pi} \int_0^{2\pi} \frac{\frac{\partial \zeta}{\partial \tau}(re^{it})}{\zeta(re^{it})} \, dt \right| \\
&\leq \frac{r}{2\pi} \int_0^{2\pi} 2|\nabla \zeta(re^{it})| \, dt \leq \frac{r}{\pi}\sqrt{2\pi} \left(\int_0^{2\pi} |\nabla \zeta(re^{it})|^2 \, dt \right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.38}
$$

On the other hand,

$$
\int_{\Omega_{R+4k,AR-4k}} |\nabla \zeta(x)|^2 \, dx = \int_{R+4k}^{AR-4k} r \int_0^{2\pi} |\nabla \zeta(re^{it})|^2 \, dt \, dr.
$$

We have $\displaystyle\int_{\Omega_{R+4k,AR-4k}} |\nabla \zeta(x)|^2 \, dx \leq E_{GL}^{\Omega_{R,AR}}(\zeta) \leq E_{GL}^{\Omega_{R,AR}}(\psi) < \varepsilon_0 \leq \frac{\pi}{8} \ln\left(\frac{AR-4k}{R+4k} \right)$ and we infer that there exists $r_* \in (R+4k, AR-4k)$ such that $\displaystyle r_* \int_0^{2\pi} |\nabla \zeta(R_* e^{it})|^2 \, dt < \frac{\pi}{8}\frac{1}{r_*}$. From (3.38) we get

$$
|deg(\zeta, \partial B(0,r_*))| < \frac{r_*}{\pi}\sqrt{2\pi} \left(\frac{\pi}{8} \frac{1}{r_*^2} \right)^{\frac{1}{2}} = \frac{1}{2}.
$$

Since the topological degree is an integer, we have necessarily $deg(\zeta, \partial B(0,r_*)) = 0$. Consequently $deg(\zeta, \partial B(0,r)) = 0$ for any $r \in (R+4k, AR-4k)$ and ζ admits a C^1 lifting $\zeta = \rho e^{i\theta}$. In fact, $\rho, \theta \in W^{2,p}_{loc}(\Omega_{R+4k,AR-4k})$ because $\zeta \in W^{2,p}_{loc}(\Omega_{R+4k,AR-4k})$ (see Theorem 3 p. 38 in [11]).

Consider $\eta_1, \eta_2 \in C^\infty(\mathbf{R})$ satisfying the following properties:

$$
\begin{array}{lll}
\eta_1 = 1 \text{ on } (-\infty, A_1], & \eta_1 = 0 \text{ on } [A_2, \infty), & \eta_1 \text{ is nonincreasing,} \\
\eta_2 = 0 \text{ on } (-\infty, A_3], & \eta_2 = 1 \text{ on } [A_4, \infty), & \eta_2 \text{ is nondecreasing.}
\end{array}
$$

Denote $\theta_0 = m(\theta, \Omega_{A_1R, A_4R})$. We define ψ_1 and ψ_2 as follows:

$$
\psi_1(x) = \begin{cases}
\psi(x) & \text{if } x \in \overline{B}(0,R), \\
\zeta(x) & \text{if } x \in B(0,A_1R) \setminus \overline{B}(0,R), \\
\left(1 + \eta_1(\frac{|x|}{R})(\rho(x)-1)\right) e^{i\left(\theta_0 + \eta_1(\frac{|x|}{R})(\theta(x)-\theta_0)\right)} & \\
\qquad\qquad\qquad \text{if } x \in B(0,A_4R) \setminus B(0,A_1R), & \\
e^{i\theta_0} & \text{if } x \in \mathbf{R}^2 \setminus B(0,A_4R),
\end{cases}
\tag{3.39}
$$

$$
\psi_2(x) = \begin{cases}
e^{i\theta_0} & \text{if } x \in \overline{B}(0,A_1R), \\
\left(1 + \eta_2(\frac{|x|}{R})(\rho(x)-1)\right) e^{i\left(\theta_0 + \eta_2(\frac{|x|}{R})(\theta(x)-\theta_0)\right)} & \\
\qquad\qquad\qquad \text{if } x \in B(0,A_4R) \setminus \overline{B}(0,A_1R), & \\
\zeta(x) & \text{if } x \in B(0,AR) \setminus B(0,A_4R), \\
\psi(x) & \text{if } x \in \mathbf{R}^2 \setminus B(0,AR).
\end{cases}
\tag{3.40}
$$

Then ψ_1, $\psi_2 \in \mathcal{E}$ and satisfy (i) and (ii). The proof of (iii), (iv) and (v) is exactly as in [43]. Next we prove (vi).

Assume that (A1) and (A2) are satisfied and let $W(s) = V(s) - V(\varphi^2(s))$. Then $W(s) = 0$ for $s \in [0, 4]$ and it is easy to see that W satisfies

$$(3.41) \qquad |W(b^2) - W(a^2)| \le C_3 |b-a| \left(a^{2p_0+1} \mathbb{1}_{\{a>2\}} + b^{2p_0+1} \mathbb{1}_{\{b>2\}} \right) \quad \text{for any } a, b \ge 0.$$

Using (1.5) and (3.41), then Hölder's inequality we obtain

$$(3.42) \qquad \begin{aligned} &\int_{\mathbf{R}^2} \left| V(|\psi|^2) - V(|\zeta|^2) \right| dx \\ &\le \int_{\Omega_{R,\,AR}} \left| V(\varphi^2(|\psi|)) - V(\varphi^2(|\zeta|)) \right| + \left| W(|\psi|^2) - W(|\zeta|^2) \right| dx \\ &\le C \int_{\Omega_{R,\,AR}} \left(\varphi^2(|\psi|) - 1 \right)^2 + \left(\varphi^2(|\zeta|) - 1 \right)^2 \, dx \\ &\quad + C \int_{\Omega_{R,\,AR}} \big| \, |\psi| - |\zeta| \, \big| \left(|\psi|^{2p_0+1} \mathbb{1}_{\{|\psi|>2\}} + |\zeta|^{2p_0+1} \mathbb{1}_{\{|\zeta|>2\}} \right) dx \\ &\le C'\varepsilon + \|\psi - \zeta\|_{L^2(\Omega_{R,\,AR})} \left[\left(\int_{\Omega_{R,\,AR}} |\psi|^{4p_0+2} \mathbb{1}_{\{|\psi|>2\}} dx \right)^{\frac{1}{2}} + \left(\int_{\Omega_{R,\,AR}} |\zeta|^{4p_0+2} \mathbb{1}_{\{|\zeta|>2\}} dx \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Using (2.2) we get

$$(3.43) \qquad \int_{\mathbf{R}^2} |\psi|^{4p_0+2} \mathbb{1}_{\{|\psi|>2\}} \, dx \le C \|\nabla \psi\|^{4p_0+2}_{L^2(\mathbf{R}^2)} \mathcal{L}^2 \left(\{ x \in \mathbf{R}^2 \mid |\psi(x)| \ge 2 \} \right).$$

On the other hand,

$$(3.44) \qquad 9\mathcal{L}^2 \left(\{ x \in \mathbf{R}^2 \mid |\psi(x)| \ge 2 \} \right) \le \int_{\mathbf{R}^2} \left(\varphi^2(|\psi|) - 1 \right)^2 \, dx \le 2E_{GL}(\psi)$$

and a similar estimate holds for ζ. We insert (3.43) and (3.44) into (3.42) to discover

$$(3.45) \qquad \int_{\mathbf{R}^2} \left| V(|\psi|^2) - V(|\zeta|^2) \right| dx \le C'\varepsilon + C\sqrt{\varepsilon} \left(E_{GL}(\psi) \right)^{p_0+1}.$$

Proceeding exactly as in [43] (see the proof of (3.88) p. 144 there) we obtain

$$(3.46) \qquad \int_{\mathbf{R}^2} \left| V(|\zeta|^2) - V(|\psi_1|) - V(|\psi_2|) \right| dx \le C\varepsilon.$$

Then (vi) follows from (3.45) and (3.46). □

Corollary 3.4 *For any $\psi \in \mathcal{E}$, there is a sequence of functions $(\psi_n)_{n\ge 1} \subset \mathcal{E}$ satisfying:*
(i) $\psi_n = \psi$ on $B(0, 2^n)$ and $\psi_n = e^{i\theta_n} = constant$ on $\mathbf{R}^N \setminus B(0, 2^{n+1})$,
(ii) $\|\nabla\psi_n - \nabla\psi\|_{L^2(\mathbf{R}^N)} \longrightarrow 0$ and $\|\varphi^2(|\psi_n|) - \varphi^2(|\psi|)\|_{L^2(\mathbf{R}^N)} \longrightarrow 0$,
(iii) $Q(\psi_n) \longrightarrow Q(\psi)$, $\displaystyle\int_{\mathbf{R}^N} \left| V(|\psi_n|^2) - V(|\psi|^2) \right| dx \longrightarrow 0$ and
$\displaystyle\int_{\mathbf{R}^N} \left| \left(\varphi^2(|\psi_n|) - 1 \right)^2 - \left(\varphi^2(|\psi|) - 1 \right)^2 \right| dx \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. Let $\varepsilon_n = E_{GL}^{\mathbf{R}^N \setminus B(0,2^n)}(\psi)$, so that $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$. Let $A = 2$, fix $1 < A_2 < A_3 < 2$ and use Lemma 3.3 with $R = 2^n$ to obtain two functions ψ_1^n, ψ_2^n with properties (i)-(vi) in that Lemma. Let $\psi_n = \psi_1^n$. It is then straightforward to prove that $(\psi_n)_{n\geq 1}$ satisfies (i)-(iii) above. $\square$

The next Lemma allows to approximate functions in $\mathcal{E}$ by functions with higher regularity.

Lemma 3.5 *(i) Assume that $\Omega = \mathbf{R}^N$ or that $\partial\Omega$ is C^1. Let $\psi \in \mathcal{E}$. For each $h > 0$, let ζ_h be a minimizer of $G_{h,\Omega}^{\psi}$ in $H_{\psi}^1(\Omega)$. Then $\|\zeta_h - \psi\|_{H^1(\Omega)} \longrightarrow 0$ as $h \longrightarrow 0$.*

(ii) Let $\psi \in \mathcal{E}$. For any $\varepsilon > 0$ and any $k \in \mathbf{N}$ there is $\zeta \in \mathcal{E}$ such that $\nabla\zeta \in H^k(\mathbf{R}^N)$, $E_{GL}(\zeta) \leq E_{GL}(\psi)$ and $\|\zeta - \psi\|_{H^1(\mathbf{R}^N)} < \varepsilon$.

Proof. (i) It suffices to prove that for any sequence $h_n \longrightarrow 0$ and any choice of a minimizer ζ_n of $G_{h_n,\Omega}^{\psi}$ in $H_{\psi}^1(\Omega)$, there is a subsequence $(\zeta_{n_k})_{k\geq 1}$ such that $\lim\limits_{k\to\infty} \|\zeta_{h_{n_k}} - \psi\|_{H^1(\Omega)} = 0$.

Let $h_n \longrightarrow 0$ and let ζ_n be as above. By (3.2) we have $\zeta_n - \psi \longrightarrow 0$ in $L^2(\Omega)$ and it is clear that $\zeta_n - \psi$ is bounded in $H_0^1(\Omega)$. Then there are $v \in H_0^1(\Omega)$ and a subsequence $(\zeta_{n_k})_{k\geq 1}$ such that

$$(\zeta_{n_k} - \psi) \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega) \quad \text{and} \quad (\zeta_{n_k} - \psi) \longrightarrow v \quad \text{a.e. on } \Omega.$$

Since $\zeta_{n_k} - \psi \longrightarrow 0$ in $L^2(\Omega)$ we infer that $v = 0$ a.e., therefore $\nabla\zeta_{n_k} \rightharpoonup \nabla\psi$ weakly in $L^2(\Omega)$ and $\zeta_{n_k} \longrightarrow \psi$ a.e on Ω. By weak convergence we have $\int_\Omega |\nabla\psi|^2\,dx \leq \liminf\limits_{k\to\infty} \int_\Omega |\nabla\zeta_{n_k}|^2\,dx$ and Fatou's Lemma gives $\int_\Omega \left(\varphi^2(|\psi|) - 1\right)^2 dx \leq \liminf\limits_{k\to\infty} \int_\Omega \left(\varphi^2(|\zeta_{n_k}|) - 1\right)^2 dx$. Thus we get $E_{GL}^{\Omega}(\psi) \leq \liminf\limits_{k\to\infty} E_{GL}^{\Omega}(\zeta_{n_k})$. On the other hand we have $E_{GL}^{\Omega}(\zeta_{n_k}) \leq E_{GL}^{\Omega}(\psi)$ for all k. We infer that necessarily $\lim\limits_{k\to\infty} E_{GL}^{\Omega}(\zeta_{n_k}) = E_{GL}^{\Omega}(\psi)$ and $\lim\limits_{k\to\infty} \int_\Omega |\nabla\zeta_{n_k}|^2\,dx = \int_\Omega |\nabla\psi|^2\,dx$. Taking into account that $\nabla\zeta_{n_k} \rightharpoonup \nabla\psi$ weakly in $L^2(\Omega)$, we deduce that $\nabla\zeta_{n_k} \longrightarrow \nabla\psi$ strongly in $L^2(\Omega)$, thus $(\zeta_{n_k} - \psi) \longrightarrow 0$ in $H_0^1(\Omega)$, as desired.

(ii) Let $h > 0$ and let ζ_h be a minimizer of $G_{h,\mathbf{R}^N}^{\psi}$. Then ζ_h satisfies (3.5) in $\mathcal{D}'(\mathbf{R}^N)$, thus $\Delta\zeta_h \in L^2(\mathbf{R}^N)$ and this implies $\frac{\partial^2 \zeta_h}{\partial x_i \partial x_j} \in L^2(\mathbf{R}^N)$ for any i, j, hence $\nabla\zeta_h \in H^1(\mathbf{R}^N)$. Moreover, if $\nabla\psi \in H^\ell(\mathbf{R}^N)$ for some $\ell \in \mathbf{N}$, taking successively the derivatives of (3.5) up to order ℓ and repeating the above argument we get $\nabla\zeta_h \in H^{\ell+1}(\mathbf{R}^N)$.

Fix $\psi \in \mathcal{E}$, $k \in \mathbf{N}$ and $\varepsilon > 0$. Using (i), there are $h_1 > 0$ and a minimizer ζ_1 of $G_{h_1,\mathbf{R}^N}^{\psi}$ such that $\|\zeta_1 - \psi\|_{H^1(\mathbf{R}^N)} < \frac{\varepsilon}{2}$ and $\nabla\zeta_1 \in H^1(\mathbf{R}^N)$. Then there are $h_2 > 0$ and a minimizer ζ_2 of $G_{h_2,\mathbf{R}^N}^{\zeta_1}$ such that $\|\zeta_2 - \zeta_1\|_{H^1(\mathbf{R}^N)} < \frac{\varepsilon}{2^2}$ and $\nabla\zeta_2 \in H^2(\mathbf{R}^N)$, and so on. After k steps we find h_k and ζ_k such that ζ_k is a minimizer of $G_{h_k,\mathbf{R}^N}^{\zeta_{k-1}}$, $\|\zeta_k - \zeta_{k-1}\|_{H^1(\mathbf{R}^N)} < \frac{\varepsilon}{2^k}$, and $\nabla\zeta_k \in H^k(\mathbf{R}^N)$. Then $\|\zeta_k - \psi\|_{H^1(\mathbf{R}^N)} < \|\zeta_k - \zeta_{k-1}\|_{H^1(\mathbf{R}^N)} + \cdots + \|\zeta_2 - \zeta_1\|_{H^1(\mathbf{R}^N)} + \|\zeta_1 - \psi\|_{H^1(\mathbf{R}^N)} < \varepsilon$. Moreover, $E_{GL}(\zeta_k) \leq E_{GL}(\zeta_{k-1}) \leq \cdots \leq E_{GL}(\psi)$. $\square$

4 Minimizing the energy at fixed momentum

The aim of this section is to investigate the existence of minimizers of the energy E under the constraint $Q = q > 0$. If such minimizers exist, they are traveling waves to (1.1) and their speed is precisely the Lagrange multiplier appearing in the variational problem.

We start with some useful properties of the functionals E, E_{GL} and Q.

Lemma 4.1 *If (A1) and (A2) in the Introduction hold, then $V(|\psi|^2) \in L^1(\mathbf{R}^N)$ whenever $\psi \in \mathcal{E}$. Moreover, for any $\delta > 0$ there exist $C_1(\delta)$, $C_2(\delta) > 0$ such that for all $\psi \in \mathcal{E}$ we have*

$$\begin{aligned}(4.1)\qquad &\frac{1-\delta}{2}\int_{\mathbf{R}^N}\left(\varphi^2(|\psi|)-1\right)^2 dx - C_1(\delta)\|\nabla\psi\|^{2^*}_{L^2(\mathbf{R}^N)} \leq \int_{\mathbf{R}^N} V(|\psi|^2)\,dx \\ &\leq \frac{1+\delta}{2}\int_{\mathbf{R}^N}\left(\varphi^2(|\psi|)-1\right)^2 dx + C_2(\delta)\|\nabla\psi\|^{2^*}_{L^2(\mathbf{R}^N)} \qquad \text{if } N \geq 3,\end{aligned}$$

respectively

$$\begin{aligned}(4.2)\qquad &\left(\frac{1-\delta}{2} - C_1(\delta)\|\nabla\psi\|^{2p_0+2}_{L^2(\mathbf{R}^2)}\right)\int_{\mathbf{R}^2}\left(\varphi^2(|\psi|)-1\right)^2 dx \leq \int_{\mathbf{R}^2} V(|\psi|^2)\,dx \\ &\leq \left(\frac{1+\delta}{2} + C_2(\delta)\|\nabla\psi\|^{2p_0+2}_{L^2(\mathbf{R}^2)}\right)\int_{\mathbf{R}^2}\left(\varphi^2(|\psi|)-1\right)^2 dx \qquad \text{if } N = 2.\end{aligned}$$

These estimates still hold if we replace the condition $F \in C^0([0,\infty))$ in (A1) by $F \in L^1_{loc}([0,\infty))$ and if we replace V by $|V|$.

Proof. Inequality (4.1) follows from Lemma 4.1 p. 144 in [43]. We only prove (4.2).

Fix $\delta > 0$. There exists $\beta = \beta(\delta) \in (0,1]$ such that

$$(4.3)\qquad \frac{1-\delta}{2}(s-1)^2 \leq V(s) \leq \frac{1+\delta}{2}(s-1)^2 \qquad \text{for any } s \in ((1-\beta)^2,(1+\beta)^2).$$

Let $\psi \in \mathcal{E}$. It follows from (4.3) that $V(|\psi|^2)\mathbb{1}_{\{1-\beta\leq|\psi|\leq 1+\beta\}} \in L^1(\mathbf{R}^2)$ and

$$\begin{aligned}(4.4)\qquad &\frac{1-\delta}{2}\int_{\{1-\beta\leq|\psi|\leq 1+\beta\}}\left(\varphi^2(|\psi|)-1\right)^2 dx \leq \int_{\{1-\beta\leq|\psi|\leq 1+\beta\}} V(|\psi|^2)\,dx \\ &\leq \frac{1+\delta}{2}\int_{\{1-\beta\leq|\psi|\leq 1+\beta\}}\left(\varphi^2(|\psi|)-1\right)^2 dx.\end{aligned}$$

Using (A2) we infer that there exists $C(\delta) > 0$ such that

$$(4.5)\qquad \left|V(s^2) - \frac{1\pm\delta}{2}\left(\varphi^2(s)-1\right)^2\right| \leq C(\delta)\left(|s-1| - \frac{1}{2}\beta\right)^{2p_0+2}$$

for any $s \geq 0$ satisfying $|s-1| \geq \beta$. Let $K = \{x \in \mathbf{R}^2 \mid \big||\psi(x)|-1\big| \geq \frac{\beta}{2}\}$. Let η be as in (3.19). Then $\left(\varphi^2(|\psi|)-1\right)^2 \geq \eta(\frac{\beta}{2})$ on K, hence

$$(4.6)\qquad \mathcal{L}^2(K) \leq \frac{1}{\eta(\frac{\beta}{2})}\int_{\mathbf{R}^2}\left(\varphi^2(|\psi|)-1\right)^2 dx.$$

Let $\tilde\psi = \left(\big||\psi|-1\big| - \frac{\beta}{2}\right)_+$. Then $\tilde\psi \in L^1_{loc}(\mathbf{R}^2)$, $|\nabla\tilde\psi| \leq |\nabla\psi|$ a.e. on $\mathbf{R}^2$ and using (2.2) we get

$$(4.7)\qquad \int_{\mathbf{R}^2}|\tilde\psi|^{2p_0+2}\,dx \leq C\|\nabla\tilde\psi\|^{2p_0+2}_{L^2(\mathbf{R}^2)}\mathcal{L}^2(K).$$

Using (4.5), (4.6) and (4.7) we obtain

$$\begin{aligned}(4.8)\qquad &\int_{\mathbf{R}^2\setminus\{1-\beta\leq|\psi|\leq 1+\beta\}}\left|V(|\psi|^2) - \frac{1\pm\delta}{2}\left(\varphi^2(|\psi|)-1\right)^2\right| dx \\ &\leq C(\delta)\int_{\mathbf{R}^2}|\tilde\psi|^{2p_0+2}\,dx \leq C'(\delta)\|\nabla\tilde\psi\|^{2p_0+2}_{L^2(\mathbf{R}^2)}\int_{\mathbf{R}^2}\left(\varphi^2(|\psi|)-1\right)^2 dx.\end{aligned}$$

From (4.4) and (4.8) we infer that $V(|\psi|^2) \in L^1(\mathbf{R}^2)$ and (4.2) holds. $\square$

The following result is a direct consequence of (4.2).

Corollary 4.2 *Assume that $N = 2$ and (A1) and (A2) hold. There is $k_1 > 0$ such that for any $\psi \in \mathcal{E}$ satisfying $\int_{\mathbf{R}^2} |\nabla\psi|^2 \, dx \le k_1$ we have $\int_{\mathbf{R}^2} V(|\psi|^2) \, dx \ge 0$.*

If $N \ge 3$ and there exists $s_0 \ge 0$ satisfying $V(s_0) < 0$, Corollary 4.2 is not valid anymore. Indeed, if V achieves negative values it easy to see that there exists $\psi \in \mathcal{E}$ such that $\int_{\mathbf{R}^N} V(|\psi|^2) \, dx < 0$. Then $\int_{\mathbf{R}^N} V(|\psi_{\sigma,\sigma}|^2) \, dx = \sigma^N \int_{\mathbf{R}^N} V(|\psi|^2) \, dx < 0$ for any $\sigma > 0$ and $\int_{\mathbf{R}^N} |\nabla\psi_{\sigma,\sigma}|^2 \, dx = \sigma^{N-2} \int_{\mathbf{R}^N} |\nabla\psi|^2 \, dx \longrightarrow 0$ as $\sigma \longrightarrow 0$.

Corollary 4.3 *Let $N \ge 2$. There is an increasing function $m : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ such that $\lim_{\tau \to 0} m(\tau) = 0$ and*

$$\| \, |\psi| - 1 \|_{L^2(\mathbf{R}^N)} \le m(E_{GL}(\psi)) \qquad \textit{for any } \psi \in \mathcal{E}.$$

Proof. Let $\tilde{F}(s) = \frac{1}{\sqrt{s}} - 1$. It is obvious that $\tilde{F}$ satisfies the assumptions (A1) and (A2) in the introduction (except the continuity at 0, but this plays no role here). Let $\tilde{V}(s) = \int_s^1 \tilde{F}(\tau) \, d\tau$, so that $\tilde{V}(s) = (\sqrt{s} - 1)^2$ and $\int_{\mathbf{R}^N} V(|\psi|^2) \, dx = \| \, |\psi| - 1 \|^2_{L^2(\mathbf{R}^N)}$. The conclusion follows by using the second inequalities in (4.1) and (4.2) with $\tilde{F}$ and $\tilde{V}$ instead of F and V. $\square$

Lemma 4.4 *(i) Let $\delta \in (0,1)$ and let $\psi \in \mathcal{E}$ be such that $1 - \delta \le |\psi| \le 1 + \delta$ a.e. on $\mathbf{R}^N$. Then*

$$|Q(\psi)| \le \frac{1}{\sqrt{2}(1-\delta)} E_{GL}(\psi).$$

(ii) Assume that $0 \le c < v_s$ and let $\varepsilon \in (0, 1 - \frac{c}{v_s})$. There exists a constant $K_1 = K_1(F, N, c, \varepsilon) > 0$ such that for any $\psi \in \mathcal{E}$ satisfying $E_{GL}(\psi) < K_1$ we have

$$\int_{\mathbf{R}^N} |\nabla\psi|^2 \, dx + \int_{\mathbf{R}^N} V(|\psi|^2) \, dx - c|Q(\psi)| \ge \varepsilon E_{GL}(\psi).$$

Proof. If $N \ge 3$, (i) is precisely Lemma 4.2 p. 145 and (ii) is Lemma 4.3 p. 146 in [43]. In the case $N = 2$ the proof is similar and is left to the reader. $\square$

For any $q \in \mathbf{R}$ we denote

$$E_{min}(q) = \inf \left\{ \int_{\mathbf{R}^N} |\nabla\psi|^2 \, dx + \int_{\mathbf{R}^N} \left| V(|\psi|^2) \right| dx \;\; \Big| \; \psi \in \mathcal{E}, Q(\psi) = q \right\}.$$

Notice that if $V \ge 0$, the above definition of E_{min} is the same as the one given in Theorem 1.1. For later purpose we need this more general definition. To simplify the notation, we denote

$$\overline{E}(\psi) = \int_{\mathbf{R}^N} |\nabla\psi|^2 \, dx + \int_{\mathbf{R}^N} \left| V(|\psi|^2) \right| dx \qquad \text{for any } \psi \in \mathcal{E}.$$

There are functions $\psi \in \mathcal{E}$ such that $Q(\psi) \ne 0$ (see for instance Lemma 4.4 p. 147 in [43]). For any $\psi \in \mathcal{E}$, the function $\tilde{\psi}(x) = \psi(-x_1, x')$ also belongs to $\mathcal{E}$ and satisfies $\overline{E}(\tilde{\psi}) = \overline{E}(\psi)$, $Q(\tilde{\psi}) = -Q(\psi)$. Taking into account (2.12), it is clear that for any q the set $\{\psi \in \mathcal{E} \mid Q(\psi) = q\}$ is not empty and $E_{min}(-q) = E_{min}(q)$. Thus it suffices to study $E_{min}(q)$ for $q \in [0, \infty)$.

If there is s_0 such that $V(s_0^2) < 0$, then $\inf\{E(\psi) \mid \psi \in \mathcal{E}, Q(\psi) = q\} = -\infty$ for all $q \in \mathbf{R}$. (This is one reason why we use $\overline{E}$, not E, in the definition of E_{min}.) Indeed, fix $q \in \mathbf{R}$. From

Corollary 3.4 and (2.12) we see that there is $\psi_* \in \mathcal{E}$ such that $Q(\psi_*) = q$ and $\psi_* = 1$ outside a ball $B(0, R_*)$. It is easy to construct a radial, real-valued function ψ_0 such that $E(\psi_0) < 0$ and $\psi_0 = 1$ outside a ball $B(0, R_0)$ (for instance, take R_0 sufficiently large, let $\psi_0 = s_0$ on $B(0, R_0 - 1)$, $\psi_0 = 1$ on $\mathbf{R}^N \setminus B(0, R_0)$ and ψ_0 affine in $|x|$ for $R_0 - 1 \le |x| \le R_0$). Then $Q(\psi_0) = 0$. Let $e_1 = (1, 0, \dots 0)$. For $n \ge 1$, we define ψ_n by $\psi_n = \psi_*$ on $B(0, R_*)$, and $\psi_n(x) = \psi_0(\frac{x}{n} - n^2(R_0 + R_*)e_1)$ on $\mathbf{R}^N \setminus B(0, R_*)$. Then $Q(\psi_n) = Q(\psi_*) + n^{N-1}Q(\psi_0) = q$ and $E(\psi_n) = E(\psi_*) + n^{N-2} \int_{\mathbf{R}^N} |\nabla \psi_0|^2 \, dx + n^N \int_{\mathbf{R}^N} V(|\psi_0|^2) \, dx \longrightarrow -\infty$ as $n \longrightarrow \infty$.

The next Lemmas establish the properties of E_{min}.

Lemma 4.5 *Assume that $N \ge 2$. For any $q > 0$ we have $E_{min}(q) \le v_s q$. Moreover, there is a sequence $(\psi_n)_{n\ge1}$ such that $\psi_n - 1 \in C_c^\infty(\mathbf{R}^N)$, $V(\psi_n) \ge 0$, $Q(\psi_n) = q$, $E(\psi_n) \longrightarrow v_s q$, $E_{GL}(\psi_n) \longrightarrow v_s q$ and $\sup\limits_{x\in\mathbf{R}^N} |\partial^\alpha \psi_n(x)| \longrightarrow 0$ as $n \longrightarrow \infty$ for any $\alpha \in \mathbf{N}^N$, $|\alpha| \ge 1$.*

Proof. Fix $\chi \in C_c^\infty(\mathbf{R}^N)$, $\chi \not\equiv 0$. We will consider three parameters ε, λ, $\sigma > 0$ such that $\varepsilon \longrightarrow 0$, $\lambda \longrightarrow \infty$, $\sigma \longrightarrow \infty$ and $\lambda \ll \sigma$. We put

$$\rho_{\varepsilon,\lambda,\sigma}(x) = 1 + \frac{\varepsilon}{\sqrt{2}\lambda} \frac{\partial \chi}{\partial x_1}\left(\frac{x_1}{\lambda}, \frac{x'}{\sigma}\right), \quad \theta_{\lambda,\sigma}(x) = \chi\left(\frac{x_1}{\lambda}, \frac{x'}{\sigma}\right), \quad \psi_{\varepsilon,\lambda,\sigma}(x) = \rho_{\varepsilon,\lambda,\sigma}(x) e^{-i\varepsilon\theta_{\lambda,\sigma}(x)}.$$

It is clear that $V(\rho^2_{\varepsilon,\lambda,\sigma}) \ge 0$ if $\frac{\varepsilon}{\lambda}$ is small enough. A straightforward computation gives

$$\int_{\mathbf{R}^N} \left|\frac{\partial \rho_{\varepsilon,\lambda,\sigma}}{\partial x_1}\right|^2 dx = \frac{\varepsilon^2 \sigma^{N-1}}{2\lambda^3} \int_{\mathbf{R}^N} \left|\frac{\partial^2 \chi}{\partial x_1^2}\right|^2 dx,$$

$$\int_{\mathbf{R}^N} \left|\frac{\partial \rho_{\varepsilon,\lambda,\sigma}}{\partial x_j}\right|^2 dx = \frac{\varepsilon^2 \sigma^{N-3}}{2\lambda} \int_{\mathbf{R}^N} \left|\frac{\partial^2 \chi}{\partial x_1 \partial x_j}\right|^2 dx, \qquad j = 2, \dots, N,$$

$$\int_{\mathbf{R}^N} \rho^2_{\varepsilon,\lambda,\sigma} \left|\frac{\partial \theta_{\lambda,\sigma}}{\partial x_1}\right|^2 dx = \frac{\sigma^{N-1}}{\lambda} \int_{\mathbf{R}^N} \left(1 + \frac{\varepsilon}{\sqrt{2}\lambda} \frac{\partial \chi}{\partial x_1}\right)^2 \left|\frac{\partial \chi}{\partial x_1}\right|^2 dx \simeq \frac{\sigma^{N-1}}{\lambda} \int_{\mathbf{R}^N} \left|\frac{\partial \chi}{\partial x_1}\right|^2 dx,$$

$$\int_{\mathbf{R}^N} \rho^2_{\varepsilon,\lambda,\sigma} \left|\frac{\partial \theta_{\lambda,\sigma}}{\partial x_j}\right|^2 dx = \sigma^{N-3}\lambda \int_{\mathbf{R}^N} \left(1 + \frac{\varepsilon}{\sqrt{2}\lambda} \frac{\partial \chi}{\partial x_1}\right)^2 \left|\frac{\partial \chi}{\partial x_j}\right|^2 dx \simeq \sigma^{N-3}\lambda \int_{\mathbf{R}^N} \left|\frac{\partial \chi}{\partial x_j}\right|^2 dx,$$

$$\int_{\mathbf{R}^N} V(\rho^2_{\varepsilon,\lambda,\sigma}) \, dx \simeq \frac{\varepsilon^2 \sigma^{N-1}}{\lambda} \int_{\mathbf{R}^N} \left|\frac{\partial \chi}{\partial x_1}\right|^2 dx,$$

$$\int_{\mathbf{R}^N} \left(\varphi^2(\rho_{\varepsilon,\lambda,\sigma}) - 1\right)^2 dx \simeq \frac{2\varepsilon^2 \sigma^{N-1}}{\lambda} \int_{\mathbf{R}^N} \left|\frac{\partial \chi}{\partial x_1}\right|^2 dx,$$

$$Q(\psi_{\varepsilon,\lambda,\sigma}) = \varepsilon \int_{\mathbf{R}^N} (\rho^2_{\varepsilon,\lambda,\sigma} - 1) \frac{\partial \theta_{\lambda,\sigma}}{\partial x_1} \, dx \simeq \frac{\sqrt{2}\varepsilon^2 \sigma^{N-1}}{\lambda} \int_{\mathbf{R}^N} \left|\frac{\partial \chi}{\partial x_1}\right|^2 dx.$$

Now fix $q > 0$. Then choose sequences of positive numbers $(\varepsilon_n)_{n\ge1}$, $(\lambda_n)_{n\ge1}$, $(\sigma_n)_{n\ge1}$ such that $\varepsilon_n \longrightarrow 0$, $\lambda_n \longrightarrow \infty$, $\sigma_n \longrightarrow \infty$, $\frac{\lambda_n}{\sigma_n} \longrightarrow 0$ and $Q(\psi_{\varepsilon_n,\lambda_n,\sigma_n}) = q$ for each n. Such a choice is possible in view of the last estimate above. In particular, this gives $\frac{\varepsilon_n^2 \sigma_n^{N-1}}{\lambda_n} \int_{\mathbf{R}^N} \left|\frac{\partial \chi}{\partial x_1}\right|^2 dx \longrightarrow \frac{q}{\sqrt{2}}$. Let $\psi_n = \psi_{\varepsilon_n,\lambda_n,\sigma_n}$. It follows from the above estimates that

$$\overline{E}(\psi_n) = E(\psi_n) = \int_{\mathbf{R}^N} |\nabla \rho_{\varepsilon,\lambda,\sigma}|^2 + \varepsilon^2 \rho^2_{\varepsilon,\lambda,\sigma} |\nabla \theta_{\lambda,\sigma}|^2 + V(\rho^2_{\varepsilon,\lambda,\sigma}) \, dx \longrightarrow \sqrt{2} q = v_s q$$

and similarly $E_{GL}(\psi_n) \longrightarrow v_s q$ as $n \longrightarrow \infty$. The other statements are obvious. Notice that a similar construction can be found in the proof of Lemma 3.3 p. 604 in [7]. $\Box$

Lemma 4.6 *Let $N \geq 2$. For each $\varepsilon > 0$ there is $q_\varepsilon > 0$ such that*

$$E_{min}(q) > (v_s - \varepsilon)q \qquad \text{for any } q \in (0, q_\varepsilon).$$

Proof. Fix $\varepsilon > 0$. It follows from Lemma 4.4 (ii) that there is $K_1(\varepsilon) > 0$ such that for any $\psi \in \mathcal{E}$ satisfying $E_{GL}(\psi) < K_1(\varepsilon)$ we have

$$\overline{E}(\psi) \geq \left(v_s - \frac{\varepsilon}{2}\right)|Q(\psi)|.$$

Using Lemma 4.1 we infer that there exists $K_2(\varepsilon) > 0$ such that for any $\psi \in \mathcal{E}$ satisfying $\overline{E}(\psi) < K_2(\varepsilon)$ we have $E_{GL}(\psi) < K_1(\varepsilon)$.

Take $q_\varepsilon = \frac{K_2(\varepsilon)}{v_s+1}$. Let $q \in (0, q_\varepsilon)$. There is $\psi \in \mathcal{E}$ such that $Q(\psi) = q$ and $\overline{E}(\psi) < E_{min}(q)+q$. Since $E_{min}(q) \leq v_s q$ by Lemma 4.5, for any such ψ we have $\overline{E}(\psi) < (v_s+1)q_\varepsilon = K_2(\varepsilon)$ and we infer that $E_{GL}(\psi) < K_1(\varepsilon)$, thus $\overline{E}(\psi) \geq \left(v_s - \frac{\varepsilon}{2}\right)|Q(\psi)| = \left(v_s - \frac{\varepsilon}{2}\right)q$. This clearly implies $E_{min}(q) \geq \left(v_s - \frac{\varepsilon}{2}\right)q$. $\square$

Lemma 4.7 *Assume that $N \geq 2$.*

(i) The function E_{min} is subadditive: for any q_1, $q_2 \geq 0$ we have $E_{min}(q_1+q_2) \leq E_{min}(q_1) + E_{min}(q_2)$.

(ii) The function E_{min} is nondecreasing on $[0,\infty)$, concave, Lipschitz continuous and its best Lipschitz constant is v_s. Moreover, for $0 < q_1 < q_2$ we have $E_{min}(q_1) \leq \left(\frac{q_1}{q_2}\right)^{\frac{N-2}{N-1}} E_{min}(q_2)$.

(iii) For any $q > 0$ we have the following alternative:

- *either $E_{min}(\tau) = v_s\tau$ for all $\tau \in [0,q]$,*
- *or $E_{min}(q) < E_{min}(\tau) + E_{min}(q-\tau)$ for all $\tau \in (0,q)$.*

Proof. (i) Fix $\varepsilon > 0$. From Corollary 3.4 and (2.12) it follows that there exist $\psi_1, \psi_2 \in \mathcal{E}$ such that $Q(\psi_i) = q_i$, $\overline{E}(\psi_i) < E_{min}(q_i) + \frac{\varepsilon}{2}$ and $\psi_i = 1$ outside a ball $B(0,R_i)$, $i = 1, 2$. Let $e \in \mathbf{R}^N$ be a vector of length 1. Define $\psi(x) = \begin{cases} \psi_1(x) & \text{if } |x| \leq R_1, \\ \psi_2(x - 4(R_1+R_2)e) & \text{otherwise.} \end{cases}$ Then $\psi \in \mathcal{E}$, $Q(\psi) = Q(\psi_1) + Q(\psi_2) = q_1 + q_2$ and $E_{min}(q_1+q_2) \leq \overline{E}(\psi) = \overline{E}(\psi_1) + \overline{E}(\psi_2) < E_{min}(q_1) + E_{min}(q_2) + \varepsilon$. Letting $\varepsilon \longrightarrow 0$ we get $E_{min}(q_1+q_2) \leq E_{min}(q_1) + E_{min}(q_2)$.

(ii) From Lemma 4.5 we obtain $0 \leq E_{min}(q) \leq v_s q$ for any $q \geq 0$.

For $\psi \in \mathcal{E}$ we have $\psi_{\sigma,\sigma} = \psi\left(\frac{\cdot}{\sigma}\right) \in \mathcal{E}$,

$$Q(\psi_{\sigma,\sigma}) = \sigma^{N-1}Q(\psi) \quad \text{and} \quad \overline{E}(\psi_{\sigma,\sigma}) = \sigma^{N-2}\int_{\mathbf{R}^N}|\nabla\psi|^2\,dx + \sigma^N\int_{\mathbf{R}^N}\left|V(|\psi|^2)\right|dx. \tag{4.9}$$

Assume that $0 < q_1 < q_2$. Let $\sigma_0 = \left(\frac{q_1}{q_2}\right)^{\frac{1}{N-1}} < 1$. For any $\psi \in \mathcal{E}$ satisfying $Q(\psi) = q_2$ we have $Q(\psi_{\sigma_0,\sigma_0}) = q_1$ and from (4.9) we see that $E_{min}(q_1) \leq \overline{E}(\psi_{\sigma_0,\sigma_0}) \leq \sigma_0^{N-2}\overline{E}(\psi)$. Passing to the infimum over all ψ verifying $Q(\psi) = q_2$ we find $E_{min}(q_1) \leq \left(\frac{q_1}{q_2}\right)^{\frac{N-2}{N-1}} E_{min}(q_2)$. In particular, E_{min} is nondecreasing. Using (i) and Lemma 4.5 we get

$$0 \leq E_{min}(q_2) - E_{min}(q_1) \leq E_{min}(q_2 - q_1) \leq v_s(q_2 - q_1).$$

Hence E_{min} is Lipschitz continuous and v_s is a Lipschitz constant for E_{min}. Lemma 4.6 implies that v_s is indeed the best Lipschitz constant of E_{min}.

Given a function f defined on $\mathbf{R}^N$ and $t \in \mathbf{R}$, we denote by $S_t^+ f$ and $S_t^- f$, respectively, the functions

$$S_t^+ f(x) = \begin{cases} f(x) & \text{if } x_N \geq t, \\ f(x_1, \dots, x_{N-1}, 2t - x_N) & \text{if } x_N < t, \end{cases} \tag{4.10}$$

$$
(4.11) \qquad S_t^- f(x) = \begin{cases} f(x_1, \dots, x_{N-1}, 2t - x_N) & \text{if } x_N \geq t, \\ f(x) \text{ if } x_N < t. \end{cases}
$$

It is easy to see that for all $\psi \in \mathcal{E}$ and $t \in \mathbf{R}$ we have $S_t^+\psi, S_t^-\psi \in \mathcal{E}$, $\overline{E}(S_t^+\psi) + \overline{E}(S_t^-\psi) = 2\overline{E}(\psi)$ and $\langle i(S_t^{\pm}\psi)_{x_1}, S_t^{\pm}\psi\rangle = S_t^{\pm}(\langle i\psi_{x_1}, \psi\rangle)$. Moreover, if $\phi \in \dot{H}^1(\mathbf{R}^N)$ then $S_t^{\pm}\phi \in \dot{H}^1(\mathbf{R}^N)$ and $\partial_{x_1}(S_t^{\pm}\phi) = S_t^{\pm}(\partial_{x_1}\phi)$. If $\psi \in \mathcal{E}$, there are $\phi \in \dot{H}^1(\mathbf{R}^N)$ and $g \in L^1(\mathbf{R}^N)$ such that $\langle i\psi_{x_1}, \psi\rangle = \partial_{x_1}\phi + g$ (see Lemma 2.1 and the remarks preceding it). Then $\langle i(S_t^{\pm}\psi)_{x_1}, S_t^{\pm}\psi\rangle = S_t^{\pm}(\langle i\psi_{x_1}, \psi\rangle) = \partial_{x_1}(S_t^{\pm}\phi) + S_t^{\pm}g$ and Definition 2.2 gives $Q(S_t^{\pm}\psi) = \int_{\mathbf{R}^N} S_t^{\pm}g\,dx$. It follows that $Q(S_t^+\psi) + Q(S_t^-\psi) = 2Q(\psi)$ and the mapping $t \longmapsto Q(S_t^+\psi) = \int_{\mathbf{R}^N} S_t^+ g\,dx = 2\int_{\{x_n \geq t\}} g\,dx$ is continuous on $\mathbf{R}$, tends to 0 as $t \longrightarrow \infty$ and to $2\int_{\mathbf{R}} g\,dx = 2Q(\psi)$ as $t \longrightarrow -\infty$.

Fix $0 < q_1 < q_2$ and $\varepsilon > 0$. Let $\psi \in \mathcal{E}$ be such that $Q(\psi) = \frac{q_1+q_2}{2}$ and $\overline{E}(\psi) < E_{min}\left(\frac{q_1+q_2}{2}\right) + \varepsilon$. The continuity of $t \longmapsto Q(S_t^+\psi)$ implies that there exists $t_0 \in \mathbf{R}$ such that $Q(S_{t_0}^+\psi) = q_1$. Then necessarily $Q(S_t^-\psi) = q_2$ and we infer that $\overline{E}(S_t^+\psi) \geq E_{min}(q_1)$, $\overline{E}(S_t^-\psi) \geq E_{min}(q_2)$, and consequently

$$
E_{min}\left(\frac{q_1+q_2}{2}\right) + \varepsilon > \overline{E}(\psi) = \frac{1}{2}(\overline{E}(S_t^+\psi) + \overline{E}(S_t^-\psi)) \geq \frac{1}{2}(E_{min}(q_1) + E_{min}(q_2)).
$$

Passing to the limit as $\varepsilon \longrightarrow 0$ in the above inequality we discover

$$
(4.12) \qquad E_{min}\left(\frac{q_1+q_2}{2}\right) \geq \frac{1}{2}(E_{min}(q_1) + E_{min}(q_2)).
$$

It is an easy exercise to prove that any continuous function satisfying (4.12) is concave.

(iii) Fix $q > 0$. By the concavity of E_{min} we have $E_{min}(\tau) \geq \frac{\tau}{q}E_{min}(q)$ for any $\tau \in (0, q)$ and equality may occur if and only if E_{min} is linear on $[0, q]$. Therefore for any $\tau \in (0, q)$ we have $E_{min}(\tau) + E_{min}(q - \tau) \geq \frac{\tau}{q}E_{min}(q) + \frac{q-\tau}{q}E_{min}(q) = E_{min}(q)$ and equality occurs if and only if E_{min} is linear on $[0, q]$, that is $E_{min}(\tau) = a\tau$ for $\tau \in [0, q]$ and some $a \in \mathbf{R}$. Then Lemma 4.5 gives $a \leq v_s$ and Lemma 4.6 implies $a \geq v_s - \varepsilon$ for any $\varepsilon > 0$, hence $a = v_s$. $\square$

The function $q \longmapsto \frac{E_{min}(q)}{q}$ is nonincreasing (because E_{min} is concave), positive and by Lemma 4.4 in [43] there is a sequence $q_n \longrightarrow \infty$ such that $\lim\limits_{n\to\infty} \frac{E_{min}(q_n)}{q_n} = 0$, hence $\lim\limits_{q\to\infty} \frac{E_{min}(q)}{q} = 0$. Let

$$
q_0 = \inf\{q > 0 \mid E_{min}(q) < v_s q\},
$$

so that $q_0 \in [0, \infty)$, $E_{min}(q) = v_s q$ for $q \in [0, q_0]$ and $E_{min}(q) < v_s q$ for any $q > q_0$.

Lemma 4.8 *Let $N \geq 2$. Assume that (A1), (A2) hold.Then for any m, $M > 0$ there exist $C_1(m)$, $C_2(M) > 0$ such that for all $\psi \in \mathcal{E}$ satisfying $m \leq \overline{E}(\psi) \leq M$ we have*

$$
C_1(m) \leq E_{GL}(\psi) \leq C_2(M).
$$

Proof. If $N \geq 3$, Lemma 4.8 follows directly from (4.1) with $|V|$ instead of V. If $N = 2$, the second inequality in (4.2) implies that there is $C_1(m) > 0$ such that $E_{GL}(\psi) \geq C_1(m)$ if $\overline{E}(\psi) \geq m$. All we have to do is to prove that $\int_{\mathbf{R}^2} \left(\varphi^2(|\psi|) - 1\right)^2 dx$ remains bounded if $\overline{E}(\psi) \leq M$. This would be trivial if $\inf\{V(s^2) \mid s \geq 0,\ |s-1| \geq \delta\} > 0$ for any $\delta > 0$; however, our assumptions do not prevent V to vanish somewhere on $[0, \infty)$ or to tend to zero at infinity. Since the proof is the same if $N = 2$ or if $N \geq 2$, let us consider the general case.

Fix $\delta \in (0, 1]$ such that $V(s^2) \geq \frac{1}{4}(s^2 - 1)^2$ for $s \in [1 - \delta, 1 + \delta]$. Consider $\psi \in \mathcal{E}$ such that $\overline{E}(\psi) \leq M$. Clearly, $\int_{\{||\psi|-1| \leq \delta\}} \left(\varphi^2(|\psi|) - 1\right)^2 dx \leq 4\int_{\{||\psi|-1| \leq \delta\}} V(|\psi|^2)\,dx \leq 4M$ and we have to prove that $\int_{\{||\psi|-1|>\delta\}} \left(\varphi^2(|\psi|) - 1\right)^2 dx$ is bounded. Since φ is bounded, it suffices to prove that $\mathcal{L}^N(\{\big||\psi|-1\big| > \delta\})$ is bounded.

Let $w = |\psi| - 1$. Then $|\nabla w| \leq |\nabla \psi|$ a.e., hence $\nabla w \in L^2(\mathbf{R}^N)$, and $\mathcal{L}^N(\{|w| > \alpha\})$ is finite for all $\alpha > 0$ (because $\psi \in \mathcal{E}$). Let $w_1(x) = \phi_1(|x|)$ and $w_2(x) = \phi_2(|x|)$ be the symmetric decreasing rearrangements of w_+ and w_-, respectively. Then φ_1 and φ_2 are finite, nonincreasing on $(0, \infty)$ and tend to zero at infinity. From Lemma 7.17 p. 174 in [38] it follows that $\|\nabla w_1\|_{L^2(\mathbf{R}^N)} \leq \|\nabla w_+\|_{L^2(\mathbf{R}^N)}$ and $\|\nabla w_2\|_{L^2(\mathbf{R}^N)} \leq \|\nabla w_-\|_{L^2(\mathbf{R}^N)}$. In particular, $w_1, w_2 \in H^1(\Omega_{R_1,R_2})$ for any $0 < R_1 < R_2 < \infty$, where $\Omega_{R_1,R_2} = B(0, R_2) \setminus \overline{B(0, R_1)}$. Using Theorem 2 p. 164 in [22] we infer that $\phi_1, \phi_2 \in H^1_{loc}((0, \infty))$, hence are continuous on $(0, \infty)$.

Let $t_i = \inf\{t \geq 0 \mid \phi_i(t) \leq \delta\}$, $i = 1, 2$, so that $0 \leq \phi_i(t) \leq \delta$ on $[t_i, \infty)$ and, if $t_i > 0$, then $\phi_i(t_i) = \delta$. It is clear that

$$
\begin{aligned}
\mathcal{L}^N(\{\big||\psi|-1\big| > \delta\}) &= \mathcal{L}^N(\{w_+ > \delta\}) + \mathcal{L}^N(\{w_- > \delta\}) \\
&= \mathcal{L}^N(\{w_1 > \delta\}) + \mathcal{L}^N(\{w_2 > \delta\}) = (t_1^N + t_2^N)\mathcal{L}^N(B(0,1)).
\end{aligned}
\tag{4.13}
$$

Define $h_1(s) = s^2 + 2s$, $H_1(s) = \frac{1}{3}s^3 + s^2$, $h_2(s) = -s^2 + 2s$, $H_2(s) = -\frac{1}{3}s^3 + s^2$, so that $H_1' = h_1$ and $H_2' = h_2$. If $t_1 > 0$ we have:

$$
\begin{aligned}
\overline{E}(\psi) &\geq \int_{\mathbf{R}^N} |\nabla \psi|^2 \, dx + \frac{1}{4} \int_{\mathbf{R}^N} \left(\varphi^2(|\psi|) - 1\right)^2 \mathbb{1}_{\{1 \leq |\psi| \leq 1+\delta\}} \, dx \\
&\geq \int_{\mathbf{R}^N} |\nabla w_+|^2 \, dx + \frac{1}{4} \int_{\{w_+ \leq \delta\}} \left((w_+ + 1)^2 - 1\right)^2 \, dx \\
&\geq \int_{\mathbf{R}^N} |\nabla w_1|^2 \, dx + \frac{1}{4} \int_{\{w_1 \leq \delta\}} \left((w_1 + 1)^2 - 1\right)^2 \, dx \\
&\geq \int_{\mathbf{R}^N \setminus B(0,t_1)} |\nabla w_1|^2 + \frac{1}{4} h_1^2(w_1) \, dx \\
&= |S^{N-1}| \int_{t_1}^{\infty} \left(|\phi_1'(s)|^2 + \frac{1}{4} h_1^2(\phi_1(s)) \right) s^{N-1} \, ds \\
&\geq t_1^{N-1} |S^{N-1}| \int_{t_1}^{\infty} |\phi_1'(s)|^2 + \frac{1}{4} h_1^2(\phi_1(s)) \, ds \\
&\geq t_1^{N-1} |S^{N-1}| \int_{t_1}^{\infty} -h_1(\phi_1(s)) \phi_1'(s) \, ds \\
&= t_1^{N-1} |S^{N-1}| \left[-H_1(\phi_1(s))\right]_{s=t_0}^{\infty} = |S^{N-1}| H_1(\delta) t_1^{N-1},
\end{aligned}
\tag{4.14}
$$

where $|S^{N-1}|$ is the surface measure of the unit sphere in $\mathbf{R}^N$. From (4.14) we get $t_1^{N-1} \leq CE(\psi)$, where C depends only on N and V. It is clear that a similar estimate holds for t_2. Then using (4.13) we obtain

$$
\mathcal{L}^N(\{\big||\psi|-1\big| > \delta\}) \leq C \left(\overline{E}(\psi)\right)^{\frac{N}{N-1}},
$$

where C depends only on N and V, and the proof of Lemma 4.8 is complete. $\square$

We can now state the main result of this section, showing precompactness of minimizing sequences for $E_{min}(q)$ as soon as $q > q_0$.

Theorem 4.9 *Assume that $q > q_0$, that is $E_{min}(q) < v_s q$. Let $(\psi_n)_{n\geq 1}$ be a sequence in $\mathcal{E}$ satisfying*

$$Q(\psi_n) \longrightarrow q \quad \text{and} \quad \overline{E}(\psi_n) \longrightarrow E_{min}(q).$$

There exist a subsequence $(\psi_{n_k})_{k\geq 1}$, a sequence of points $(x_k)_{k\geq 1} \subset \mathbf{R}^N$, and $\psi \in \mathcal{E}$ such that $Q(\psi)=q$, $\overline{E}(\psi)=E_{min}(q)$, $\psi_{n_k}(\cdot + x_k) \longrightarrow \psi$ a.e. on $\mathbf{R}^N$ and $d_0(\psi_{n_k}(\cdot + x_k), \psi) \longrightarrow 0$, that is

$$\|\nabla\psi_{n_k}(\cdot + x_k) - \nabla\psi\|_{L^2(\mathbf{R}^N)} \longrightarrow 0, \qquad \|\,|\psi_{n_k}|(\cdot + x_k) - |\psi|\,\|_{L^2(\mathbf{R}^N)} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty.$$

Proof. Since $\overline{E}(\psi_n) \longrightarrow E_{min}(q) > 0$, it follows from Lemma 4.8 that there are two positive constants M_1, M_2 such that $M_1 \leq E_{GL}(\psi_n) \leq M_2$ for all sufficiently large n. Passing to a subsequence if necessary, we may assume that $E_{GL}(\psi_n) \longrightarrow \alpha_0 > 0$.

We will use the concentration-compactness principle [39]. We denote by $\Lambda_n(t)$ the concentration function associated to $E_{GL}(\psi_n)$, that is

$$\Lambda_n(t) = \sup_{y\in\mathbf{R}^N} \int_{B(y,t)} |\nabla\psi_n|^2 + \frac{1}{2}\left(\varphi^2(|\psi_n|) - 1\right)^2 \, dx. \tag{4.15}$$

Proceeding as in [39], it is straightforward to prove that there exists a subsequence of $((\psi_n, \Lambda_n))_{n\geq 1}$, still denoted $((\psi_n, \Lambda_n))_{n\geq 1}$, there exists a nondecreasing function $\Lambda : [0,\infty) \longrightarrow \mathbf{R}$ and there is $\alpha \in [0, \alpha_0]$ such that

$$\Lambda_n(t) \longrightarrow \Lambda(t) \text{ a.e on } [0,\infty) \text{ as } n \longrightarrow \infty \qquad \text{and} \qquad \Lambda(t) \longrightarrow \alpha \text{ as } t \longrightarrow \infty. \tag{4.16}$$

As in the proof of Theorem 5.3 in [43], we see that there is a nondecreasing sequence $t_n \longrightarrow \infty$ such that

$$\lim_{n\to\infty} \Lambda_n(t_n) = \lim_{n\to\infty} \Lambda_n\left(\frac{t_n}{2}\right) = \alpha. \tag{4.17}$$

Our aim is to prove that $\alpha = \alpha_0$. The next lemma implies that $\alpha > 0$.

Lemma 4.10 *Assume that $N \geq 2$ and assumptions (A1) and (A2) in the Introduction hold. Let $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ be a sequence satisfying:*

(a) $E_{GL}(\psi_n) \leq M$ for some positive constant M.

(b) $\liminf_{n\to\infty} Q(\psi_n) \geq q \in \mathbf{R} \cup \{\infty\}$ as $n \longrightarrow \infty$.

c) $\limsup_{n\to\infty} E(\psi_n) < v_s q$.

Then there exists $k > 0$ such that $\sup_{y\in\mathbf{R}^N} E_{GL}^{B(y,1)}(\psi_n) \geq k$ for all sufficiently large n.

Proof. We argue by contradiction and we suppose that the conclusion is false. Then there is a subsequence (still denoted $(\psi_n)_{n\geq 1}$) such that

$$\lim_{n\to\infty} \sup_{y\in\mathbf{R}^N} \int_{B(y,1)} |\nabla\psi_n|^2 + \frac{1}{2}\left(\varphi^2(|\psi_n|) - 1\right)^2 \, dx = 0. \tag{4.18}$$

The first step is to prove that

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \left| V(|\psi_n|^2) - \frac{1}{2}\left(\varphi^2(|\psi_n|) - 1\right)^2 \right| dx = 0. \tag{4.19}$$

If $N \geq 3$ this is done exactly as in the proof of Lemma 5.4 p. 156 in [43]. We consider here only the case $N = 2$.

Fix $\varepsilon > 0$. By (A1) there is $\delta(\varepsilon) > 0$ such that

$$\left| V(s^2) - \frac{1}{2}\left(\varphi^2(s) - 1\right)^2 \right| \le \frac{\varepsilon}{2}\left(\varphi^2(s) - 1\right)^2 \qquad \text{for any } s \in [1 - \delta(\varepsilon), 1 + \delta(\varepsilon)],$$

hence

$$\begin{aligned} &\int_{\{1-\delta(\varepsilon)\le|\psi|\le 1+\delta(\varepsilon)\}} \left| V(|\psi_n|^2) - \frac{1}{2}\left(\varphi^2(|\psi_n|) - 1\right)^2 \right| dx \\ &\le \frac{\varepsilon}{2}\int_{\{1-\delta(\varepsilon)\le|\psi|\le 1+\delta(\varepsilon)\}} \left(\varphi^2(|\psi_n|) - 1\right)^2 dx \le \varepsilon M. \end{aligned} \tag{4.20}$$

Using (A2) we infer that there is $C(\varepsilon) > 0$ such that

$$\left| V(s^2) - \frac{1}{2}\left(\varphi^2(s) - 1\right)^2 \right| \le C(\varepsilon)(|s| - 1)^{2p_0+2} \qquad \text{for any } s \text{ satisfying } |s - 1| \ge \delta(\varepsilon). \tag{4.21}$$

Let $w_n = \big| |\psi_n| - 1 \big|$. Then $w_n \in L^1_{loc}(\mathbf{R}^N)$ and $|\nabla w_n| \le |\nabla \psi_n|$ a.e., hence $\|\nabla w_n\|_{L^2(\mathbf{R}^2)} \le \|\nabla \psi_n\|_{L^2(\mathbf{R}^2)} \le \sqrt{M}$. Using (2.2) for $\left(w_n - \frac{\delta(\varepsilon)}{2}\right)_+$ we obtain

$$\begin{aligned} &\int_{\{w_n > \delta(\varepsilon)\}} w_n^{2p_0+2}\, dx \le 2^{2p_0+2} \int_{\{w_n > \delta(\varepsilon)\}} \left(w_n - \frac{\delta(\varepsilon)}{2}\right)_+^{2p_0+2} dx \\ &\le C\|\nabla w_n\|^{2p_0+2}_{L^2(\mathbf{R}^2)} \mathcal{L}^2(\{w_n > \tfrac{\delta(\varepsilon)}{2}\}) \le CM^{p_0+1}\mathcal{L}^2(\{w_n > \tfrac{\delta(\varepsilon)}{2}\}). \end{aligned} \tag{4.22}$$

We claim that for any $\delta > 0$ we have

$$\lim_{n\to\infty} \mathcal{L}^2(\{w_n > \delta\}) = 0. \tag{4.23}$$

The proof of (4.23) relies on Lieb's Lemma (see Lemma 6 p. 447 in [37] or Lemma 2.2 p. 101 in [12]) and is the same as the proof of (5.20) p. 157 in [43], so we omit it.

From (4.21), (4.22) and (4.23) we get

$$\int_{\{||\psi|-1|>\delta(\varepsilon)\}} \left| V(s^2) - \frac{1}{2}\left(\varphi^2(s) - 1\right)^2 \right| dx \le C(\varepsilon) \int_{\{w_n > \delta(\varepsilon)\}} w_n^{2p_0+2}\, dx \longrightarrow 0 \tag{4.24}$$

as $n \longrightarrow \infty$. Then (4.19) follows from (4.20) and (4.24).

From (4.18) and Lemma 3.2 we infer that there exists a sequence $h_n \longrightarrow 0$ and for each n there is a minimizer ζ_n of $G^{\psi_n}_{h_n,\mathbf{R}^N}$ in $H^1_{\psi_n}(\mathbf{R}^N)$ such that

$$\delta_n := \| \, |\zeta_n| - 1\|_{L^\infty(\mathbf{R}^N)} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{4.25}$$

Then Lemma 4.4 (i) implies

$$E_{GL}(\zeta_n) \ge \sqrt{2}(1 - \delta_n)|Q(\zeta_n)|. \tag{4.26}$$

From (3.4) we obtain $\lim\limits_{n\to\infty} |Q(\zeta_n) - Q(\psi_n)| = 0$, hence $\liminf\limits_{n\to\infty} Q(\zeta_n) = \liminf\limits_{n\to\infty} Q(\psi_n) \ge q$. Using (4.19), the fact that $E_{GL}(\zeta_n) \le E_{GL}(\psi_n)$ and (4.26) we get

$$\begin{aligned} E(\psi_n) &= E_{GL}(\psi_n) + \int_{\mathbf{R}^N} V(|\psi_n|^2) - \frac{1}{2}(\varphi^2(|\psi_n|) - 1)^2\, dx \\ &\ge E_{GL}(\zeta_n) + \int_{\mathbf{R}^N} V(|\psi_n|^2) - \frac{1}{2}(\varphi^2(|\psi_n|) - 1)^2\, dx \\ &\ge \sqrt{2}(1 - \delta_n)|Q(\zeta_n)| + \int_{\mathbf{R}^N} V(|\psi_n|^2) - \frac{1}{2}(\varphi^2(|\psi_n|) - 1)^2\, dx. \end{aligned}$$

Passing to the limit as $n \longrightarrow \infty$ in the above inequality we get

$$\liminf_{n\to\infty} E(\psi_n) \geq \sqrt{2}q = v_s q,$$

which contradicts assumption c) in Lemma 4.10. This ends the proof of Lemma 4.10. $\square$

Next we prove that $\alpha \notin (0, \alpha_0)$. We argue again by contradiction and we assume that $0 < \alpha < \alpha_0$. Let t_n be as in (4.17) and let $R_n = \frac{t_n}{2}$. For each $n \geq 1$, fix $y_n \in \mathbf{R}^N$ such that $E_{GL}^{B(y_n,R_n)}(\psi_n) \geq \Lambda_n(R_n) - \frac{1}{n}$. Using (4.17), we have

$$\varepsilon_n := E_{GL}^{B(y_n,2R_n)\setminus B(y_n,R_n)}(\psi_n) \leq \Lambda_n(2R_n) - \left(\Lambda_n(R_n) - \frac{1}{n}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{4.27}$$

After a translation, we may assume that $y_n = 0$. Using Lemma 3.3 with $A = 2$, $R = R_n$, $\varepsilon = \varepsilon_n$, we infer that for all n sufficiently large there exist two functions $\psi_{n,1}$, $\psi_{n,2}$ having the properties (i)-(vi) in Lemma 3.3. In particular, we have $E_{GL}(\psi_{n,1}) \geq E_{GL}^{B(0,R_n)}(\psi_n) \geq q(R_n) - \frac{1}{n}$, $E_{GL}(\psi_{n,2}) \geq E_{GL}^{\mathbf{R}^N\setminus B(0,2R_n)}(\psi_n) \geq E_{GL}(\psi_n) - q(2R_n)$ and $|E_{GL}(\psi_n) - E_{GL}(\psi_{n,1}) - E_{GL}(\psi_{n,2})| \leq C\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$. Taking into account (4.17), we conclude that necessarily

$$E_{GL}(\psi_{n,1}) \longrightarrow \alpha \qquad \text{and} \qquad E_{GL}(\psi_{n,2}) \longrightarrow \alpha_0 - \alpha \qquad \text{as } n \longrightarrow \infty. \tag{4.28}$$

From Lemma 3.3 (iii)-(vi) we get

$$|\overline{E}(\psi_n) - \overline{E}(\psi_{n,1}) - \overline{E}(\psi_{n,2})| \longrightarrow 0 \qquad \text{and} \tag{4.29}$$

$$|Q(\psi_n) - Q(\psi_{n,1}) - Q(\psi_{n,2})| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{4.30}$$

In particular, $\overline{E}(\psi_{n,i})$ is bounded, $i = 1, 2$. Passing to a subsequence if necessary, we may assume that $\overline{E}(\psi_{n,i}) \longrightarrow m_i \geq 0$ as $n \longrightarrow \infty$. Since $\lim\limits_{n\to\infty} E_{GL}(\psi_{n,i}) > 0$, it follows from Lemma 4.1 that $m_i > 0$, $i = 1, 2$. Using (4.29) we see that $m_1 + m_2 = E_{min}(q)$, hence $m_1, m_2 \in (0, E_{min}(q))$.

Assume that $\liminf\limits_{n\to\infty} Q(\psi_{n,1}) \leq 0$. Then (4.30) implies $\limsup\limits_{n\to\infty} Q(\psi_{n,2}) \geq q$. It is obvious that

$$\overline{E}(\psi_{n,2}) \geq E_{min}(Q(\psi_{n,2})).$$

Passing to lim sup in the above inequality and using the continuity and the monotonicity of E_{min} we get $m_2 \geq E_{min}(q)$, a contradiction. Thus necessarily $\liminf\limits_{n\to\infty} Q(\psi_{n,1}) > 0$ and similarly $\liminf\limits_{n\to\infty} Q(\psi_{n,2}) > 0$. From (4.30) we get $\limsup\limits_{n\to\infty} Q(\psi_{n,i}) < q$, $i = 1, 2$. Passing again to a subsequence, we may assume that $Q(\psi_{n,i}) \longrightarrow q_i$ as $n \longrightarrow \infty$, $i = 1, 2$, where $q_1, q_2 \in (0, q)$. Using (4.30) we infer that $q_1 + q_2 = q$. Since $\overline{E}(\psi_{n,i}) \geq E_{min}(Q(\psi_{n,i}))$, passing to the limit we get $m_i \geq E_{min}(q_i)$, $i = 1, 2$ and consequently

$$E_{min}(q) = m_1 + m_2 \geq E_{min}(q_1) + E_{min}(q_2).$$

Since $E_{min}(q) < v_s q$, the above inequality is in contradiction with Lemma 4.7 (iii). Thus we cannot have $\alpha \in (0, \alpha_0)$.

So far we have proved that $\alpha = \alpha_0$. Then it is standard to prove that there is a sequence $(x_n)_{n\geq 1} \subset \mathbf{R}^N$ such that for any $\varepsilon > 0$ there is $R_\varepsilon > 0$ satisfying $E_{GL}^{\mathbf{R}^N\setminus B(x_n,R_\varepsilon)}(\psi_n) < \varepsilon$ for all sufficiently large n. Denoting $\tilde{\psi}_n = \psi_n(\cdot + x_n)$, we see that for any $\varepsilon > 0$ there exist $R_\varepsilon > 0$ and $n_\varepsilon \in \mathbf{N}$ such that

$$E_{GL}^{\mathbf{R}^N\setminus B(0,R_\varepsilon)}(\tilde{\psi}_n) < \varepsilon \qquad \text{for all } n \geq n_\varepsilon. \tag{4.31}$$

Obviously, $(\nabla\tilde{\psi}_n)_{n\geq 1}$ is bounded in $L^2(\mathbf{R}^N)$ and it is easy to see that $(\tilde{\psi}_n)_{n\geq 1}$ is bounded in $L^2(B(0,R))$ for any $R>0$ (use (2.3) and (2.4) if $N=2$, respectively (2.3) and the Sobolev embedding if $N\geq 3$). By a standard argument, there exist a function $\psi \in H^1_{loc}(\mathbf{R}^N)$ such that $\nabla\psi \in L^2(\mathbf{R}^N)$ and a subsequence $(\tilde{\psi}_{n_k})_{k\geq 1}$ satisfying

$$(4.32)\qquad \begin{array}{ll} \nabla\tilde{\psi}_{n_k} \rightharpoonup \nabla\psi & \text{weakly in } L^2(\mathbf{R}^N), \\ \tilde{\psi}_{n_k} \rightharpoonup \psi & \text{weakly in } H^1(B(0,R)) \text{ for all } R>0, \\ \tilde{\psi}_{n_k} \longrightarrow \psi & \text{strongly in } L^p(B(0,R)) \text{ for } R>0 \text{ and } p\in[1,2^*)\ (p\in[1,\infty) \text{ if } N=2), \\ \tilde{\psi}_{n_k} \longrightarrow \psi & \text{a.e. on } \mathbf{R}^N. \end{array}$$

By weak convergence we have

$$(4.33)\qquad \int_{\mathbf{R}^N} |\nabla\psi|^2\,dx \leq \liminf_{k\to\infty}\int_{\mathbf{R}^N} |\nabla\tilde{\psi}_{n_k}|^2\,dx.$$

The a.e. convergence and Fatou's Lemma imply

$$(4.34)\qquad \int_{\mathbf{R}^N} \left(\varphi^2(|\psi|)-1\right)^2 dx \leq \liminf_{k\to\infty}\int_{\mathbf{R}^N} \left(\varphi^2(|\tilde{\psi}_{n_k}|)-1\right)^2 dx \quad \text{and}$$

$$(4.35)\qquad \int_{\mathbf{R}^N} \left|V(|\psi|^2)\right| dx \leq \liminf_{k\to\infty}\int_{\mathbf{R}^N} \left|V(|\tilde{\psi}_{n_k}|^2)\right| dx.$$

From (4.33), (4.34) and (4.35) we obtain

$$(4.36)\qquad E_{GL}(\psi) \leq \liminf_{k\to\infty} E_{GL}(\tilde{\psi}_{n_k}) = \alpha_0 \quad \text{and} \quad \overline{E}(\psi) \leq \liminf_{k\to\infty} \overline{E}(\tilde{\psi}_{n_k}) = E_{min}(q).$$

Similarly, for any $\varepsilon>0$ we get

$$(4.37)\qquad E_{GL}^{\mathbf{R}^N\setminus B(0,R_\varepsilon)}(\psi) \leq \liminf_{k\to\infty} E_{GL}^{\mathbf{R}^N\setminus B(0,R_\varepsilon)}(\tilde{\psi}_{n_k}) \leq \limsup_{k\to\infty} E_{GL}^{\mathbf{R}^N\setminus B(0,R_\varepsilon)}(\tilde{\psi}_{n_k}) \leq \varepsilon.$$

The following holds.

Lemma 4.11 *Assume that $N\geq 2$ and assumptions (A1) and (A2) are verified. Let $(\gamma_n)_{n\geq 1}\subset \mathcal{E}$ be a sequence satisfying:*

(a) $(E_{GL}(\gamma_n))_{n\geq 1}$ is bounded and for any $\varepsilon>0$ there are $R_\varepsilon>0$ and $n_\varepsilon\in\mathbf{N}$ such that $E_{GL}^{\mathbf{R}^N\setminus B(0,R_\varepsilon)}(\gamma_n)<\varepsilon$ for $n\geq n_\varepsilon$.

(b) There exists $\gamma\in\mathcal{E}$ such that $\gamma_n\longrightarrow\gamma$ strongly in $L^2(B(0,R))$ for any $R>0$, and $\gamma_n\longrightarrow\gamma$ a.e. on $\mathbf{R}^N$ as $n\longrightarrow\infty$.

Then $\|\,|\gamma_n|-|\gamma|\,\|_{L^2(\mathbf{R}^N)}\longrightarrow 0$ and $\|V(|\gamma_n|^2)-V(|\gamma|^2)\|_{L^1(\mathbf{R}^N)}\longrightarrow 0$ as $n\longrightarrow\infty$.

Proof of Lemma 4.11. Fix $\varepsilon>0$. Let R_ε and $n_\varepsilon\in\mathbf{N}$ be as in assumption (a). Then

$$(4.38)\qquad \|\varphi(|\gamma_n|)-1\|^2_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} \leq \int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \left(\varphi^2(|\gamma_n|)-1\right)^2 dx \leq 2\varepsilon$$

for $n\geq n_\varepsilon$. It is clear that a similar estimate holds for γ. Let

$$\begin{array}{ll} \tilde{\gamma}_n = |\gamma_n| - \varphi(|\gamma_n|), & \tilde{\gamma} = |\gamma| - \varphi(|\gamma|), \\ A_n = \{x\in\mathbf{R}^N \mid |\gamma_n(x)|\geq 2\}, & A = \{x\in\mathbf{R}^N \mid |\gamma(x)|\geq 2\}, \\ A_n^\varepsilon = \{x\in\mathbf{R}^N\setminus B(0,R_\varepsilon) \mid |\gamma_n(x)|\geq 2\}, & A^\varepsilon = \{x\in\mathbf{R}^N\setminus B(0,R_\varepsilon) \mid |\gamma(x)|\geq 2\}. \end{array}$$

We have

$$9\mathcal{L}^N(A_n^\varepsilon) \le \int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \left(\varphi^2(|\gamma_n|)-1\right)^2 dx \le 2E_{GL}^{\mathbf{R}^N\setminus B(0,R_\varepsilon)}(\gamma_n) \le 2\varepsilon$$

and similarly $9\mathcal{L}^N(A^\varepsilon) \le 2\varepsilon$. In the same way $\mathcal{L}^N(A_n) \le \frac{2}{9}E_{GL}(\gamma_n)$ and $\mathcal{L}^N(A) \le \frac{2}{9}E_{GL}(\gamma)$.

Since $0 \le \varphi' \le 1$, it is easy to see that $|\nabla\tilde{\gamma}_n| \le |\nabla\gamma_n|$ a.e. and $|\nabla\tilde{\gamma}| \le |\nabla\gamma|$ a.e., hence $(|\nabla\tilde{\gamma}_n|)_{n\ge1}$ and $\nabla\tilde{\gamma}$ are bounded in $L^2(\mathbf{R}^N)$. If $N \ge 3$, the Sobolev embedding implies that $(\tilde{\gamma}_n)_{n\ge1}$ is bounded in $L^{2^*}(\mathbf{R}^N)$. Then using the fact that $\tilde{\gamma}_n = 0$ on $\mathbf{R}^N \setminus A_n$ and Hölder's inequality we infer that $\tilde{\gamma}_n$ is bounded in $L^p(\mathbf{R}^N)$ for $1 \le p \le 2^*$. If $N = 2$, by (2.2) we get

$$\|\tilde{\gamma}_n\|_{L^p(\mathbf{R}^2)}^p \le C_p^p \|\nabla\tilde{\gamma}_n\|_{L^2(\mathbf{R}^2)}^p \mathcal{L}^2(A_n),$$

hence $(\tilde{\gamma}_n)_{n\ge1}$ is bounded in $L^p(\mathbf{R}^N)$ for any $2 \le p < \infty$. Let $p = 2^*$ if $N \ge 3$ and let $p > 2p_0+2$ if $N = 2$. Using Hölder's inequality ($p > 2p_0 + 2 > 2$) we have

$$\|\tilde{\gamma}_n\|_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))}^2 = \int_{A_n^\varepsilon} |\tilde{\gamma}_n|^2\,dx \le \|\tilde{\gamma}_n\|_{L^p(\mathbf{R}^N)}^2 \mathcal{L}^N(A_n^\varepsilon)^{1-\frac{2}{p}} \le C_1\varepsilon^{1-\frac{2}{p}}, \tag{4.39}$$

where C_1 does not depend on n. It is clear that a similar estimate holds for $\tilde{\gamma}$.

In the same way, using (A2) and Hölder's inequality ($p > 2p_0 + 2$) we get

$$\begin{aligned}
&\int_{(\mathbf{R}^N\setminus B(0,R_\varepsilon))\cap\{|\gamma_n|\ge4\}} |V(|\gamma_n|^2)|\,dx \le C' \int_{(\mathbf{R}^N\setminus B(0,R_\varepsilon))\cap\{|\gamma_n|\ge4\}} |\gamma_n|^{2p_0+2}\,dx \\
&\le C'' \int_{A_n^\varepsilon} |\tilde{\gamma}_n|^{2p_0+2}\,dx \le C''\|\tilde{\gamma}_n\|_{L^p(\mathbf{R}^N)}^{2p_0+2}\mathcal{L}^N(A_n^\varepsilon)^{1-\frac{2p_0+2}{p}} \le C_2\varepsilon^{1-\frac{2p_0+2}{p}},
\end{aligned} \tag{4.40}$$

and (A1) implies

$$\int_{(\mathbf{R}^N\setminus B(0,R_\varepsilon))\cap\{|\gamma_n|\le4\}} |V(|\gamma_n|^2)|\,dx \le C''' \int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} \left(\varphi^2(|\gamma_n|)-1\right)^2 dx \le C_3\varepsilon, \tag{4.41}$$

where the constants C_2, C_3 do not depend on n. The same estimates are obviously valid for γ.

From (4.38) and (4.39) we get

$$\begin{aligned}
\|\,|\gamma_n|-|\gamma|\,\|_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} &\le \|\varphi(|\gamma_n|)-1\|_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} + \|\varphi(|\gamma|)-1\|_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} \\
&+ \|\tilde{\gamma}_n\|_{L^2(\mathbf{R}^N\setminus B(0,R_\varepsilon))} + \|\tilde{\gamma}\|_{L^2(\mathbf{R}^N\setminus B(0,\mathbf{R}_\varepsilon))} \le 2\sqrt{2}\varepsilon + 2C_1\varepsilon^{1-\frac{2}{p}}.
\end{aligned} \tag{4.42}$$

Using (4.40) and (4.41) we obtain

$$\int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)} |V(|\gamma_n|^2)|\,dx \le C_2\varepsilon^{1-\frac{2p_0+2}{p}} + C_3\varepsilon. \tag{4.43}$$

It is obvious that γ also satisfies (4.43).

Since $|\gamma_n| = \varphi(|\gamma_n|) + \tilde{\gamma}_n$ is bounded in $L^p(B(0,R))$ for any $p \in [2, 2^*]$ if $N \ge 3$, respectively $p \in [2,\infty)$ if $N = 2$, and $\gamma_n \longrightarrow \gamma$ in $L^2(B(0,R))$ by assumption (b), using interpolation we infer that $\gamma_n \longrightarrow \gamma$ in $L^p(B(0,R))$ for any $p \in [1, 2^*)$ (with $2^* = \infty$ if $N = 2$). This implies that $V(|\gamma_n|^2) \longrightarrow V(|\gamma|^2)$ in $L^1(B(0,R))$ (see, for instance, Theorem A2 p. 133 in [46]). Thus we have $\|\,|\gamma_n| - |\gamma|\,\|_{L^2(B(0,R_\varepsilon))} \le \varepsilon$ and $\|V(|\gamma_n|^2) - V(|\gamma|^2)\|_{L^1(B(0,R_\varepsilon))} \le \varepsilon$ for all sufficiently large n. Together with inequalities (4.42) and (4.43), this implies $\|\,|\gamma_n|-|\gamma|\,\|_{L^2(\mathbf{R}^N)} \le 2\sqrt{2}\varepsilon + 2C_1\varepsilon^{1-\frac{2}{p}} + \varepsilon$ and $\|V(|\gamma_n|^2) - V(|\gamma|^2)\|_{L^1(\mathbf{R}^N)} \le 2C_2\varepsilon^{1-\frac{2p_0+2}{p}} + (2C_3+1)\varepsilon$ for all sufficiently large n. Since ε is arbitrary, Lemma 4.11 follows. □

We come back to the proof of Theorem 4.9. From (4.31), (4.32) and Lemma 4.11 we obtain $\|\,|\tilde{\psi}_{n_k}| - |\psi|\,\|_{L^2(\mathbf{R}^N)} \longrightarrow 0$ as $k \longrightarrow \infty$. Clearly, this implies $\|\varphi^2(|\tilde{\psi}_{n_k}|) - \varphi^2(|\psi|)\|_{L^2(\mathbf{R}^N)} \longrightarrow 0$.

We will use the following result:

Lemma 4.12 *Let $N \geq 2$ and assume that $(\gamma_n)_{n\geq 1} \subset \mathcal{E}$ is a sequence satisfying:*

(a) $(E_{GL}(\gamma_n))_{n\geq 1}$ is bounded and for any $\varepsilon > 0$ there are $R_\varepsilon > 0$ and $n_\varepsilon \in \mathbf{N}$ such that $E_{GL}^{\mathbf{R}^N \setminus B(0,R_\varepsilon)}(\gamma_n) < \varepsilon$ for $n \geq n_\varepsilon$.

(b) There exists $\gamma \in \mathcal{E}$ such that $\nabla \gamma_n \rightharpoonup \nabla \gamma$ weakly in $L^2(\mathbf{R}^N)$ and $\gamma_n \longrightarrow \gamma$ strongly in $L^2(B(0,R))$ for any $R > 0$ as $n \longrightarrow \infty$.

Then $Q(\gamma_n) \longrightarrow Q(\gamma)$ as $n \longrightarrow \infty$.

We postpone the proof of Lemma 4.12 and we complete the proof of Theorem 4.9. From (4.31), (4.32) and Lemma 4.12 it follows that $Q(\psi) = \lim_{k\to\infty} Q(\tilde{\psi}_{n_k}) = q$. Then necessarily $\overline{E}(\psi) \geq E_{min}(q) = \lim_{k\to\infty} \overline{E}(\tilde{\psi}_{n_k})$. From (4.36) we get $\overline{E}(\psi) = E_{min}(q)$, hence ψ is a minimizer of $\overline{E}$ under the constraint $Q(\psi) = q$. Taking into account (4.33), (4.35) and the fact that $\overline{E}(\tilde{\psi}_{n_k}) \longrightarrow \overline{E}(\psi)$, we infer that $\int_{\mathbf{R}^N} |\nabla \tilde{\psi}_{n_k}|^2\, dx \longrightarrow \int_{\mathbf{R}^N} |\nabla \psi|^2\, dx$. Together with the weak convergence $\nabla \tilde{\psi}_{n_k} \rightharpoonup \nabla \psi$ in $L^2(\mathbf{R}^N)$, this gives the strong convergence $\|\nabla \tilde{\psi}_{n_k} - \nabla \psi\|_{L^2(\mathbf{R}^N)} \longrightarrow 0$ as $k \longrightarrow \infty$ and Theorem 4.9 is proven. $\Box$

Proof of Lemma 4.12. It follows from Lemma 4.1 and Lemma 4.4 (ii) that there are $\varepsilon_0 > 0$ and $C_0 > 0$ such that for any $\phi \in \mathcal{E}$ satisfying $E_{GL}(\phi) \leq \varepsilon_0$ we have

$$|Q(\phi)| \leq C_0 E_{GL}(\phi). \tag{4.44}$$

Fix $\varepsilon \in (0, \frac{\varepsilon_0}{2})$. Let R_ε and n_ε be as in assumption (a). We will use the conformal transform. Let

$$v_k(x) = \begin{cases} \gamma_k(x) & \text{if } |x| \geq R_\varepsilon, \\ \gamma_k\left(\frac{R_\varepsilon^2}{|x|^2} x\right) & \text{if } |x| < R_\varepsilon, \end{cases} \qquad v(x) = \begin{cases} \gamma(x) & \text{if } |x| \geq R_\varepsilon, \\ \gamma\left(\frac{R_\varepsilon^2}{|x|^2} x\right) & \text{if } |x| < R_\varepsilon. \end{cases} \tag{4.45}$$

A straightforward computation gives

$$\int_{B(0,R_\varepsilon)} |\nabla v_k|^2\, dx = \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |\nabla \gamma_k(y)|^2 \left(\frac{R_\varepsilon^2}{|y|^2}\right)^{N-2} dy \leq \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} |\nabla \gamma_k(y)|^2\, dy, \tag{4.46}$$

$$\begin{aligned} \int_{B(0,R_\varepsilon)} \left(\varphi^2(|v_k|) - 1\right)^2\, dx &= \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} \left(\varphi^2(|\gamma_k(y)|) - 1\right)^2 \left(\frac{R_\varepsilon^2}{|y|^2}\right)^{N} dy \\ &\leq \int_{\mathbf{R}^N \setminus B(0,R_\varepsilon)} \left(\varphi^2(|\gamma_k|) - 1\right)^2\, dy, \end{aligned} \tag{4.47}$$

so that $v_k \in \mathcal{E}$ and $E_{GL}(v_k) < 2\varepsilon < \varepsilon_0$. Similarly $v \in \mathcal{E}$ and $E_{GL}(v) < 2\varepsilon$. From (4.44) we get

$$|Q(v_k)| \leq 2C_0\varepsilon \qquad \text{and} \qquad |Q(v)| \leq 2C_0\varepsilon. \tag{4.48}$$

Since $\nabla \gamma_k \rightharpoonup \nabla \gamma$ weakly in $L^2(\mathbf{R}^N)$, a simple change of variables shows that for any fixed $\delta \in (0, R_\varepsilon)$ we have $\nabla v_k \rightharpoonup \nabla v$ weakly in $L^2(B(0,R_\varepsilon) \setminus B(0,\delta))$. On the other hand,

$$\int_{B(0,\delta)} |\nabla v_k|^2\, dx = \int_{\mathbf{R}^N \setminus B(0,\frac{R_\varepsilon^2}{\delta})} |\nabla \gamma_k(y)|^2 \left(\frac{R_\varepsilon^2}{|y|^2}\right)^{N-2} dy \leq \int_{\mathbf{R}^N \setminus B(0,\frac{R_\varepsilon^2}{\delta})} |\nabla \gamma_k(y)|^2\, dy$$

and $\sup_{k\geq 1} \int_{\mathbf{R}^N \setminus B(0,\frac{R_\varepsilon^2}{\delta})} |\nabla \gamma_k(y)|^2\, dy \longrightarrow 0$ as $\delta \longrightarrow 0$ by assumption (a). We conclude that

$$\nabla v_k \rightharpoonup \nabla v \qquad \text{weakly in } L^2(B(0,R_\varepsilon)). \tag{4.49}$$

Since $\gamma_k \longrightarrow \gamma$ in $L^2(B(0,R))$ for any $R > 0$, we have for any fixed $\delta \in (0, R_\varepsilon)$,

$$\int_{B(0,R_\varepsilon)\setminus B(0,\delta)} |v_k - v|^2 \, dx = \int_{B(0,\frac{R_\varepsilon^2}{\delta})\setminus B(0,R_\varepsilon)} |\gamma_k(y) - \gamma(y)|^2 \left(\frac{R_\varepsilon^2}{|y|^2} \right)^N dy \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

It is easy to see that there is $p > 2$ such that $\left((|v_k| - 2)_+\right)_{k\geq 1}$ is bounded in $L^p(\mathbf{R}^N)$. (If $N \geq 3$ this follows for $p = 2^*$ from the Sobolev embedding because $\|\nabla v_k\|^2_{L^2(\mathbf{R}^N)} \leq E_{GL}(v_k) \leq 2\varepsilon$. If $N = 2$, the fact that $E_{GL}(v_k) \leq 2\varepsilon$ implies that $\mathcal{L}^2(\{|v_k| \geq 2\})$ and $\|\nabla v_k\|_{L^2(\mathbf{R}^2)}$ are bounded and the conclusion follows from (2.2).) Using Hölder's inequality we obtain

$$\int_{B(0,\delta)} (|v_k| - 2)_+^2 \, dx \leq \| \left(|v_k| - 2\right)_+ \|^2_{L^p(\mathbf{R}^N)} \left(\mathcal{L}^N(B(0,\delta))\right)^{1-\frac{2}{p}}$$

and the last quantity tends to zero as $\delta \longrightarrow 0$ uniformly with respect to k. This implies

$$\int_{B(0,\delta)} |v_k|^2 \, dx \longrightarrow 0 \qquad \text{as } \delta \longrightarrow 0 \text{ uniformly with respect to } k$$

and we conclude that

$$v_k \longrightarrow v \qquad \text{in } L^2(B(0,R_\varepsilon)). \tag{4.50}$$

Let

$$w_k = \gamma_k - v_k, \qquad w = \gamma - v. \tag{4.51}$$

It is obvious that w_k, $w \in H_0^1(B(0,R_\varepsilon))$, $\gamma_k = v_k + w_k$ and $\gamma = v + w$. From assumption (b), (4.49) and (4.50) it follows that

$$w_k \longrightarrow w \quad \text{strongly} \quad \text{and} \quad \nabla w_k \rightharpoonup \nabla w \quad \text{weakly in } L^2(B(0,R_\varepsilon)). \tag{4.52}$$

Using Definition 2.2 we have

$$\begin{aligned} &|Q(\gamma_k) - Q(\gamma)| \leq |Q(v_k) - Q(v)| + \left|L(\langle i\tfrac{\partial v_k}{\partial x_1}, w_k\rangle - \langle i\tfrac{\partial v}{\partial x_1}, w\rangle)\right| \\ &+\left|L(\langle i\tfrac{\partial w_k}{\partial x_1}, v_k\rangle - \langle i\tfrac{\partial w}{\partial x_1}, v\rangle)\right| + \left|L(\langle i\tfrac{\partial w_k}{\partial x_1}, w_k\rangle - \langle i\tfrac{\partial w}{\partial x_1}, w\rangle)\right|. \end{aligned} \tag{4.53}$$

From (4.48) we get $|Q(v_k) - Q(v)| \leq 4C_0\varepsilon$. Since $w_k = 0$ and $w = 0$ outside $\overline{B}(0, R_\varepsilon)$, using the definition of L we obtain

$$L(\langle i\tfrac{\partial v_k}{\partial x_1}, w_k\rangle - \langle i\tfrac{\partial v}{\partial x_1}, w\rangle) = \int_{B(0,R_\varepsilon)} \langle i\tfrac{\partial v_k}{\partial x_1} - i\tfrac{\partial v}{\partial x_1}, w\rangle + \langle i\tfrac{\partial v_k}{\partial x_1}, w_k - w\rangle \, dx \longrightarrow 0 \quad \text{ as } k \longrightarrow \infty$$

because $\frac{\partial v_k}{\partial x_1} - \frac{\partial v}{\partial x_1} \rightharpoonup 0$ weakly and $w_k - w \longrightarrow 0$ strongly in $L^2(B(0,R_\varepsilon))$. Similarly the last two terms in (4.53) tend to zero as $k \longrightarrow \infty$. Finally we get $|Q(\gamma_k) - Q(\gamma)| \leq (4C_0 + 1)\varepsilon$ for all sufficiently large k. Since $\varepsilon \in (0, \frac{\varepsilon_0}{2})$ is arbitrary, the conclusion of Lemma 4.12 follows. $\square$

Corollary 4.13 *Assume that $N \geq 2$ and (A1), (A2) are satisfied. If $(\gamma_n)_{n\geq 1} \subset \mathcal{E}$, $\gamma \in \mathcal{E}$ are such that $d_0(\gamma_n, \gamma) \longrightarrow 0$, then $\lim\limits_{n\to\infty} Q(\gamma_n) = Q(\gamma)$ and $\lim\limits_{n\to\infty} \|V(|\gamma_n|^2) - V(|\gamma|^2)\|_{L^1(\mathbf{R}^N)} = 0$.*

In particular, Q and E are continuous functionals on $\mathcal{E}$ endowed with the semi-distance d_0.

Proof. We have $\nabla\gamma_n \longrightarrow \nabla\gamma$ and $(|\gamma_n| - |\gamma|) \longrightarrow 0$ in $L^2(\mathbf{R}^N)$ as $n \longrightarrow \infty$, hence $|\nabla\gamma_n|^2 + \frac{1}{2}\left(\varphi^2(|\gamma_n|) - 1\right)^2 \longrightarrow |\nabla\gamma|^2 + \frac{1}{2}\left(\varphi^2(|\gamma|) - 1\right)^2$ in $L^1(\mathbf{R}^N)$, and consequently $(\gamma_n)_{n\geq 1}$ satisfies assumption (a) in Lemma 4.12.

Consider a subsequence $(\gamma_{n_\ell})_{\ell\geq 1}$ of $(\gamma_n)_{n\geq 1}$. Then there exist a subsequence $(\gamma_{n_{\ell_k}})_{k\geq 1}$ and $\gamma_0 \in \mathcal{E}$ that satisfy (4.32). Since $\nabla\gamma_{n_{\ell_k}} \rightharpoonup \nabla\gamma_0$ weakly in $L^2(\mathbf{R}^N)$ and $\nabla\gamma_{n_{\ell_k}} \longrightarrow \nabla\gamma$ in $L^2(\mathbf{R}^N)$ we see that $\nabla\gamma_0 = \nabla\gamma$ a.e. on $\mathbf{R}^N$, hence there is a constant $\beta \in \mathbf{C}$ such that $\gamma_0 = \gamma + \beta$ a.e. on $\mathbf{R}^N$. The convergence $|\gamma_{n_{\ell_k}}| \longrightarrow |\gamma_0|$ in $L^2_{loc}(\mathbf{R}^N)$ gives $|\gamma_0| = |\gamma|$ a.e. on $\mathbf{R}^N$. By the definition of Q it follows that $Q(\gamma_0) = Q(\gamma + \beta) = Q(\gamma)$. Using Lemma 4.12 we get $Q(\gamma_{n_{\ell_k}}) \longrightarrow Q(\gamma_0) = Q(\gamma)$ as $k \longrightarrow \infty$ and Lemma 4.11 implies that $V(|\gamma_{n_{\ell_k}}|^2) \longrightarrow V(|\gamma_0|^2) = V(|\gamma|^2)$ in $L^1(\mathbf{R}^N)$ as $k \longrightarrow \infty$. Hence any subsequence $(\gamma_{n_\ell})_{\ell\geq 1}$ of $(\gamma_n)_{n\geq 1}$ contains a subsequence $(\gamma_{n_{\ell_k}})_{k\geq 1}$ such that $Q(\gamma_{n_{\ell_k}}) \longrightarrow Q(\gamma)$ and $\|V(|\gamma_{n_{\ell_k}}|^2) - V(|\gamma|^2)\|_{L^1(\mathbf{R}^N)} \longrightarrow 0$, and this clearly implies the desired conclusion. $\square$

Assume that for some $q > 0$ there is $\psi \in \mathcal{E}$ such that $Q(\psi) = q$ and $\overline{E}(\psi) = E_{min}(q)$. Using Corollary 4.13, for any sequence $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ such that $d_0(\psi_n, \psi) \longrightarrow 0$ and for any sequence of points $(x_n)_{n\geq 1} \subset \mathbf{R}^N$ we have $Q(\psi_n(\cdot + x_n)) \longrightarrow q$ and $\overline{E}(\psi_n(\cdot + x_n)) \longrightarrow E_{min}(q)$. Hence the convergence result provided by Theorem 4.9 for minimizing (sub)sequences of $\overline{E}$ under the constraint $Q = q$ is optimal.

Next we show that if $V \geq 0$ on $[0, \infty)$, the minimizers of $\overline{E} = E$ at fixed momentum are traveling waves to (1.1). We denote by $d^-E_{min}(q)$ and $d^+E_{min}(q)$ the left and right derivatives of E_{min} at $q > 0$ (which exist and are finite for any $q > 0$ because E_{min} is concave). We have:

Proposition 4.14 *Let $N \geq 2$ and $q > 0$. Assume that $V(s) \geq 0$ for any $s \geq 0$ and ψ is a minimizer of E in the set $\{\phi \in \mathcal{E} \mid Q(\phi) = q\}$. Then:*

(i) There is $c \in [d^+E_{min}(q), d^-E_{min}(q)]$ such that ψ satisfies

$$
ic\psi_{x_1} + \Delta\psi + F(|\psi|^2)\psi = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{4.54}
$$

(ii) Any solution $\psi \in \mathcal{E}$ of (4.54) satisfies $\psi \in W^{2,p}_{loc}(\mathbf{R}^N)$ and $\nabla\psi \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$, ψ and $\nabla\psi$ are bounded and $\psi \in C^{1,\alpha}(\mathbf{R}^N)$ for any $\alpha \in [0, 1)$.

(iii) After a translation, ψ is axially symmetric with respect to the x_1-axis if $N \geq 3$. The same conclusion holds for $N = 2$ if we assume in addition that F is C^1.

(iv) For any $q > q_0$ there are $\psi^+, \psi^- \in \mathcal{E}$ such that $Q(\psi^+) = Q(\psi^-) = p$, $E(\psi^+) = E(\psi^-) = E_{min}(p)$ and ψ^+, ψ^- satisfy (4.54) with speeds $c^+ = d^+E_{min}(p)$ and $c^- = d^-E_{min}(p)$, respectively.

Proof. (i) It is easy to see that $\Delta\psi + F(|\psi|^2)\psi \in H^{-1}(\mathbf{R}^N)$, $i\psi_{x_1} \in L^2(\mathbf{R}^N)$ and for any $\phi \in C^\infty_c(\mathbf{R}^N)$ we have $\psi + \phi \in \mathcal{E}$, $\lim\limits_{t\to 0} \frac{1}{t}(Q(\psi + t\phi) - Q(\psi)) = 2\langle i\psi_{x_1}, \phi\rangle_{L^2(\mathbf{R}^N)}$ and

$$
\begin{aligned}
\lim_{t\to 0} \frac{1}{t}(E(\psi + t\phi) - E(\psi)) &= 2\int_{\mathbf{R}^N} \langle\nabla\psi, \nabla\phi\rangle - F(|\psi|^2)\langle\psi, \phi\rangle \, dx \\
&= -2\langle\Delta\psi + F(|\psi|^2)\psi, \phi\rangle_{H^{-1}(\mathbf{R}^N), H^1(\mathbf{R}^N)}.
\end{aligned} \tag{4.55}
$$

Denote $E'(\psi).\phi = -2\langle\Delta\psi + F(|\psi|^2)\psi, \phi\rangle_{H^{-1}(\mathbf{R}^N), H^1(\mathbf{R}^N)}$ and $Q'(\psi).\phi = 2\langle i\psi_{x_1}, \phi\rangle_{L^2(\mathbf{R}^N)}$. We have $E(\psi + t\phi) \geq E_{min}(Q(\psi + t\phi))$, hence for all $t > 0$

$$
\frac{1}{t}(E(\psi + t\phi) - E(\psi)) \geq \frac{1}{t}(E_{min}(Q(\psi + t\phi)) - E_{min}(q)). \tag{4.56}
$$

If $Q'(\psi).\phi > 0$, we have $Q(\psi + t\phi) > Q(\psi) = q$ for $t > 0$ and t close to 0, then passing to the limit as $t \downarrow 0$ in (4.56) we get $E'(\psi).\phi \geq d^+E_{min}(q)Q'(\psi).\phi$. If $Q'(\psi).\phi < 0$, we have

$Q(\psi+t\phi) < Q(\psi) = q$ for t close to 0 and $t>0$, then passing to the limit as $t \downarrow 0$ in (4.56) we get $E'(\psi).\phi \geq d^- E_{min}(q) Q'(\psi).\phi$. Putting $-\phi$ instead of ϕ in the above, we discover

$$(4.57) \qquad \begin{array}{ll} d^+ E_{min}(q) Q'(\psi).\phi \leq E'(\psi).\phi \leq d^- E_{min}(q) Q'(\psi).\phi & \text{if } Q'(\psi).\phi > 0, \text{ and} \\ d^- E_{min}(q) Q'(\psi).\phi \leq E'(\psi).\phi \leq d^+ E_{min}(q) Q'(\psi).\phi & \text{if } Q'(\psi).\phi < 0. \end{array}$$

Let $\phi_0 \in C_c^\infty(\mathbf{R}^N)$ be such that $Q'(\psi).\phi_0 = 0$. We claim that $E'(\psi).\phi_0 = 0$. To see this, consider $\phi \in C_c^\infty(\mathbf{R}^N)$ such that $Q'(\psi).\phi \neq 0$. (Such a ϕ exists for otherwise, we would have $0 = Q'(\psi).\phi = 2\langle i\psi_{x_1}, \phi\rangle_{L^2(\mathbf{R}^N)}$ for any $\phi \in C_c^\infty(\mathbf{R}^N)$, yielding $\psi_{x_1} = 0$, hence $Q(\psi) = 0 \neq q$.) Then for any $n \in \mathbf{N}$ we have $Q'(\psi).(\phi + n\phi_0) = Q'(\psi).\phi$. From (4.57) it follows that $E'(\psi).(\phi + n\phi_0) = E'(\psi).\phi + nE'(\psi).\phi_0$ is bounded, thus necessarily $E'(\psi).\phi_0 = 0$.

Take $\phi_1 \in C_c^\infty(\mathbf{R}^N)$ such that $Q'(\psi).\phi_1 = 1$. Let $c = E'(\psi).\phi_1$. Using (4.57) we obtain $c \in [d^+E_{min}(q), d^-E_{min}(q)]$. For any $\phi \in C_c^\infty(\mathbf{R}^N)$ we have $Q'(\psi).(\phi - (Q'(\psi).\phi)\phi_1) = 0$, hence $E'(\psi).(\phi - (Q'(\psi).\phi)\phi_1) = 0$, that is $E'(\psi).\phi = cQ'(\psi).\phi$ and this is precisely (4.54).

(ii) If $N \geq 3$ this is Lemma 5.5 in [43]. If $N = 2$ the proof is very similar and we omit it.

(iii) If $N \geq 3$, the axial symmetry follows from the fact that the minimizers are C^1 and from Theorem 2' p. 329 in [42]. We use an argument due to O. Lopes [40] to give a proof which requires F to be C^1, but works also for $N = 2$. Let S_t^+ and S_t^- be as in (4.10) and (4.11), respectively. Proceeding as in the proof of Lemma 4.7 (ii), we find $t \in \mathbf{R}$ such that $Q(S_t^+\psi) = Q(S_t^-\psi) = q$. This implies $E(S_t^+\psi) \geq E_{min}(q)$ and $E(S_t^-\psi) \geq E_{min}(q)$. On the other hand $E(S_t^+\psi) + E(S_t^-\psi) = 2E(\psi) = 2E_{min}(q)$, thus necessarily $E(S_t^+\psi) = E(S_t^-\psi) = E_{min}(q)$ and $S_t^+\psi$ and $S_t^-\psi$ are also minimizers. Then $S_t^+\psi$ and $S_t^-\psi$ satisfy (4.54) (with some coefficients c_+ and c_- instead of c) and have the regularity properties given by (ii). Since $S_t^+\psi = \psi$ on $\{x_N > t\}$ and $S_t^-\psi = \psi$ on $\{x_N < t\}$, we infer that necessarily $c_+ = c_- = c$. Let $\phi_0(x) = e^{\frac{icx_1}{2}}\psi(x)$, $\phi_1(x) = e^{\frac{icx_1}{2}}S_t^+\psi(x)$, $\phi_2(x) = e^{\frac{icx_1}{2}}S_t^-\psi(x)$. Then ϕ_0, ϕ_1 and ϕ_2 are bounded, belong to $W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [2,\infty)$ and solve the equation

$$\Delta\phi + \left(\frac{c^2}{4} + F(|\phi|^2)\right)\phi = 0 \qquad \text{in } \mathbf{R}^N.$$

Since F is C^1 and ϕ_0, ϕ_1 are bounded, the function $w = \phi_1 - \phi_0$ satisfies an equation

$$\Delta w + A(x)w = 0 \qquad \text{in } \mathbf{R}^N,$$

where $A(x)$ is a 2×2 matrix and $A \in L^\infty(\mathbf{R}^N)$. Since $w \in H^2_{loc}(\mathbf{R}^N)$ and $w = 0$ in $\{x_N > t\}$, the Unique Continuation Theorem (see, for instance, the appendix of [40]) implies that $w = 0$ on $\mathbf{R}^N$, that is $S_t^+\psi = \psi$ on $\mathbf{R}^N$. We have thus proved that ψ is symmetric with respect to the hyperplane $\{x_N = t\}$. Similarly we prove that for any $e \in S^{N-1}$ orthogonal to $e_1 = (1,0,\ldots,0)$ there is $t_e \in \mathbf{R}$ such that ψ is symmetric with respect to the hyperplane $\{x \in \mathbf{R}^N \mid x.e = t_e\}$. Then it is easy to see that after a translation ψ is symmetric with respect to Ox_1.

iv) Consider a sequence $q_n \uparrow q$. We may assume $q_n > q_0$ for each n. By Theorem 4.9 there is $\psi_n \in \mathcal{E}$ such that $Q(\psi_n) = q_n \longrightarrow q$ and $E(\psi_n) = E_{min}(q_n) \longrightarrow E_{min}(q)$ by continuity of E_{min}. Since $q > q_0$ we have $E_{min}(q) < v_s q$ and using Theorem 4.9 again we infer that there are a subsequence $(\psi_{n_k})_{k\geq 1}$, a sequence $(x_k)_{k\geq 1} \subset \mathbf{R}^N$ and $\psi^- \in \mathcal{E}$ such that $Q(\psi^-) = q$, $E(\psi^-) = E_{min}(q)$ and, denoting $\tilde{\psi}_{n_k} = \psi_{n_k}(\cdot + x_n)$, we have $\tilde{\psi}_{n_k} \longrightarrow \psi^-$ a.e. on $\mathbf{R}^N$ and $d_0(\tilde{\psi}_{n_k}, \psi^-) \longrightarrow 0$ as $k \longrightarrow \infty$.

By (i) we know that each $\tilde{\psi}_{n_k}$ satisfies (4.54) for some $c_{n_k} \in [d^+E_{min}(q_{n_k}), d^-E_{min}(q_{n_k})]$. Since E_{min} is concave, we have $c_{n_k} \longrightarrow d^-E_{min}(q)$ as $k \longrightarrow \infty$. It is easily seen that $\tilde{\psi}_{n_k} \longrightarrow \psi^-$ and $F(|\tilde{\psi}_{n_k}|^2)\tilde{\psi}_{n_k} \longrightarrow F(|\psi^-|^2)\psi^-$ in $\mathcal{D}'(\mathbf{R}^N)$. Writing (4.54) for each $\tilde{\psi}_{n_k}$ and passing to the limit as $k \longrightarrow \infty$ we infer that ψ^- satisfies (4.54) in $\mathcal{D}'(\mathbf{R}^N)$ with $c = d^-E_{min}(q)$.

The same argument for a sequence $q_n \downarrow q$ gives the existence of ψ^+. $\square$

If F satisfies assumption (A4) in the introduction and $F''(1) \neq 3$, we prove that in space dimension $N = 2$ we have $q_0 = 0$. This implies that we can minimize E under the constraint $Q = q$ for any $q > 0$. The traveling waves obtained in this way have small energy and speed tending to v_s as $q \longrightarrow 0$. For the two-dimensional Gross-Pitaevskii equation, the numerical and formal study in [31] suggests that these traveling waves are rarefaction pulses asymptotically described by the ground states of the Kadomtsev-Petviashvili I (KP-I) equation. The rigorous convergence, up to rescaling and renormalization, of the traveling waves of (1.1) in the transonic limit to the ground states of the (KP-I) equation has been proven in [6] in the case of the two-dimensional Gross-Pitaevskii equation. That result has been extended in [16] to a general nonlinearity satisfying (A1), (A2) and (A4) with $F''(1) \neq 3$.

A result similar to Theorem 4.15 below is *not* true in higher dimensions: in view of Proposition 1.4 we have $q_0 > 0$ for any $N \geq 3$. If $N \geq 3$, the existence of traveling waves with speed close to v_s is guaranteed by Theorem 1.1 and Corollary 1.2 p. 113 in [43]. In space dimension three, the convergence of the traveling waves constructed in [43] to the ground states of the three-dimensional (KP-I) equation as $c \to v_s$ has been rigorously justified under the same assumptions as in dimension two (see Theorem 6 in [16]). It was also shown in [16] that these solutions have high energy and momentum (of order $1/\sqrt{v_s^2 - c^2}$ as $c \to v_s$) and thus lie on the upper branch in figure 1 (b).

Theorem 4.15 *Suppose that $N = 2$, the assumption (A4) in the introduction holds and $F''(1) \neq 3$. Then $E_{min}(q) < v_s q$ for any $q > 0$. In other words, $q_0 = 0$.*

Remark 4.16 If $N = 2$, $V \geq 0$ and (A1), (A2) and (A4) hold with $F''(1) \neq 3$, it follows from Theorems 4.15 and 4.9 that for any $q > 0$ there is $\psi_q \in \mathcal{E}$ such that $Q(\psi_q) = q$ and $E(\psi_p) = E_{min}(q)$. Proposition 4.14 (i) implies that ψ_q is a traveling wave of (1.1) of speed $c(\psi_q) \in [d^+E_{min}(q), d^-E_{min}(q)]$. Using Lemmas 4.5 and 4.6 we infer that $c(\psi_q) \longrightarrow v_s$ as $q \longrightarrow 0$. In particular, we see that there are traveling waves of arbitrarily small energy whose speeds are arbitrarily close to v_s.

In view of the formal asymptotics given in [31], it is natural to try to prove Theorem 4.15 by using test functions constructed from an ansatz related to the (KP-I) equation.

Proof of Theorem 4.15. Fix $\gamma > 0$ (to be chosen later). We consider the (KP-I) equation

$$u_t - \gamma u u_x + \frac{1}{v_s^2} u_{xxx} - \partial_x^{-1} u_{yy} = 0, \qquad t \in \mathbf{R}, (x,y) \in \mathbf{R}^2, \tag{4.58}$$

where u is real-valued. Let Y be the completion of $\{\partial_x \phi \mid \phi \in C_c^\infty(\mathbf{R}^2, \mathbf{R})\}$ for the norm $\|\partial_x \phi\|_Y^2 = \|\partial_x \phi\|_{L^2(\mathbf{R}^2)}^2 + v_s^2 \|\partial_y \phi\|_{L^2(\mathbf{R}^2)}^2 + \|\partial_{xx} \phi\|_{L^2(\mathbf{R}^2)}^2$. A traveling wave for (4.58) moving with velocity $\frac{1}{v_s^2}$ is a solution of the form $u(t,x,y) = v(x - \frac{t}{v_s^2}, y)$, where $v \in Y$. The traveling wave profile v solves the equation

$$\frac{1}{v_s^2} v_x + \gamma v v_x - \frac{1}{v_s^2} v_{xxx} + \partial_x^{-1} v_{yy} = 0 \qquad \text{in } \mathbf{R}^2,$$

or equivalently, after integrating in x,

$$\frac{1}{v_s^2} v + \frac{\gamma}{2} v^2 - \frac{1}{v_s^2} v_{xx} + \partial_x^{-2} v_{yy} = 0 \qquad \text{in } \mathbf{R}^2. \tag{4.59}$$

It is a critical point of the functional (called the *action*)

$$\mathscr{S}(v) = \int_{\mathbf{R}^2} \frac{1}{v_s^2} |v|^2 + \frac{1}{v_s^2} |v_x|^2 + |\partial_x^{-1} v_y|^2 \, dx \, dy + \frac{\gamma}{3} \int_{\mathbf{R}^2} v^3 \, dx \, dy = \frac{1}{v_s^2} \|v\|_Y^2 + \frac{\gamma}{3} \int_{\mathbf{R}^2} v^3 \, dx \, dy.$$

Equation (4.59) is indeed nonlinear if $\gamma \neq 0$. The existence of a nontrivial traveling wave solution w for (KP-I) follows from Theorem 3.1 p. 217 in [19]. The solution found in [19] minimizes $\|\cdot\|_Y$ in the set $\{v \in Y \mid \int_{\mathbf{R}^2} v^3\,dx\,dy = \int_{\mathbf{R}^2} w^3\,dx\,dy\}$. It was also proved (see Theorem 4.1 p. 227 in [19]) that $w \in H^\infty(\mathbf{R}^2) := \cap_{m\in\mathbf{N}} H^m(\mathbf{R}^2)$, $\partial_x^{-1} w_y \in H^\infty(\mathbf{R}^2)$ and w minimizes the action $\mathscr{S}$ among all nontrivial solutions of (4.59) (that is, w is a *ground state*). Moreover, w satisfies the following integral identities:

$$\left\{\begin{array}{l} \displaystyle\int_{\mathbf{R}^2} \frac{1}{v_s^2} w^2 + \frac{\gamma}{2} w^3 + \frac{1}{v_s^2}|\partial_x w|^2 + |\partial_x^{-1} w_y|^2\,dx\,dy = 0, \\[2ex] \displaystyle\int_{\mathbf{R}^2} \frac{1}{v_s^2} w^2 + \frac{\gamma}{3} w^3 - \frac{1}{v_s^2}|\partial_x w|^2 + 3|\partial_x^{-1} w_y|^2\,dx\,dy = 0, \\[2ex] \displaystyle\int_{\mathbf{R}^2} \frac{1}{v_s^2} w^2 + \frac{\gamma}{3} w^3 + \frac{1}{v_s^2}|\partial_x w|^2 - |\partial_x^{-1} w_y|^2\,dx\,dy = 0. \end{array}\right. \tag{4.60}$$

The first identity is obtained by multiplying (4.59) by w and integrating, while the two other are Pohozaev identities associated to the scalings in x, respectively in y. They are formally obtained by multiplying (4.59) by xw, respectively by $y\partial_x^{-1} w_y$ and integrating by parts; see the proof of Theorem 1.1 p. 214 in [19] for a rigorous justification.

Comparing $\mathscr{S}(w)$ to the last equality in (4.60) we get

$$\int_{\mathbf{R}^2} |\partial_x^{-1} w_y|^2\,dx\,dy = \frac{1}{2}\mathscr{S}(w). \tag{4.61}$$

In particular, $\mathscr{S}(w) > 0$. Then from the three identities (4.60) we obtain

$$\frac{1}{v_s^2}\int_{\mathbf{R}^2} |w|^2\,dx\,dy = \frac{3}{2}\mathscr{S}(w), \qquad \frac{1}{v_s^2}\int_{\mathbf{R}^2} |w_x|^2\,dx\,dy = \mathscr{S}(w), \qquad \frac{\gamma}{6}\int_{\mathbf{R}^2} w^3\,dx\,dy = -\mathscr{S}(w). \tag{4.62}$$

Let w be as above and let $\phi = v_s \partial_x^{-1} w$, so that $\partial_x \phi = v_s w$. For $\varepsilon > 0$ small we define

$$\rho_\varepsilon(x,y) = 1 + \varepsilon^2 w(\varepsilon x, \varepsilon^2 y), \qquad \theta_\varepsilon(x,y) = \varepsilon\phi(\varepsilon x, \varepsilon^2 y), \qquad U_\varepsilon = \rho_\varepsilon e^{-i\theta_\varepsilon}.$$

Then $U_\varepsilon \in \mathcal{E}$ (because $w \in H^\infty(\mathbf{R}^2)$). For ε sufficiently small we have $V(|U_\varepsilon|^2) = V(\rho_\varepsilon^2) \geq 0$, hence $\overline{E}(U_\varepsilon) = E(U_\varepsilon)$. A straightforward computation and (4.61), (4.62) give

$$\int_{\mathbf{R}^2} \Big|\frac{\partial \rho_\varepsilon}{\partial x}\Big|^2 dx\,dy = \varepsilon^3 \int_{\mathbf{R}^2} \Big|\frac{\partial w}{\partial x}\Big|^2 dx\,dy = \varepsilon^3 v_s^2 \mathscr{S}(w) = 2\varepsilon^3 \mathscr{S}(w),$$

$$\int_{\mathbf{R}^2} \Big|\frac{\partial \rho_\varepsilon}{\partial y}\Big|^2 dx\,dy = \varepsilon^5 \int_{\mathbf{R}^2} \Big|\frac{\partial w}{\partial y}\Big|^2 dx\,dy,$$

$$\begin{aligned} &\int_{\mathbf{R}^2} \rho_\varepsilon^2 \Big|\frac{\partial \theta_\varepsilon}{\partial x}\Big|^2 dx\,dy = \varepsilon \int_{\mathbf{R}^2} (1+\varepsilon^2 w)^2 |\phi_x|^2 dx\,dy \\ &= \varepsilon v_s^2 \int_{\mathbf{R}^2} (1+\varepsilon^2 w)^2 w^2 dx\,dy = \frac{3}{2} v_s^4 \mathscr{S}(w)\varepsilon - \frac{12}{\gamma} v_s^2 \mathscr{S}(w)\varepsilon^3 + v_s^2 \varepsilon^5 \int_{\mathbf{R}^2} w^4 dx\,dy, \end{aligned}$$

$$\begin{aligned} &\int_{\mathbf{R}^2} \rho_\varepsilon^2 \Big|\frac{\partial \theta_\varepsilon}{\partial y}\Big|^2 dx\,dy = \varepsilon^3 \int_{\mathbf{R}^2} (1+\varepsilon^2 w)^2 |\phi_y|^2 dx\,dy \\ &= \varepsilon^3 v_s^2 \int_{\mathbf{R}^2} (1+\varepsilon^2 w)^2 |\partial_x^{-1} w_y|^2 dx\,dy \\ &= \frac{1}{2} v_s^2 \mathscr{S}(w)\varepsilon^3 + 2\varepsilon^5 v_s^2 \int_{\mathbf{R}^2} w|\partial_x^{-1} w_y|^2 dx\,dy + \varepsilon^7 v_s^2 \int_{\mathbf{R}^2} w^2 |\partial_x^{-1} w_y|^2 dx\,dy. \end{aligned}$$

Using (2.7) we get

$$
\begin{aligned}
Q(U_\varepsilon) &= \int_{\mathbf{R}^2} (\rho_\varepsilon^2 - 1)\frac{\partial \theta_\varepsilon}{\partial x} dx\, dy = \varepsilon \int_{\mathbf{R}^2} (2w + \varepsilon^2 w^2)\phi_x \, dx\, dy \\
&= \varepsilon v_s \int_{\mathbf{R}^2} (2w + \varepsilon^2 w^2) w \, dx\, dy = 3 v_s^3 \mathscr{S}(w)\varepsilon - \frac{6}{\gamma} v_s \mathscr{S}(w)\varepsilon^3.
\end{aligned} \tag{4.63}
$$

If (A4) holds we have the expansion

$$
V(s) = \frac{1}{2}(s-1)^2 - \frac{1}{6}F''(1)(s-1)^3 + H(s), \tag{4.64}
$$

where $|H(s)| \le C(s-1)^4$ for s close to 1. Using (4.64) and the fact that $w \in L^p(\mathbf{R}^2)$ for any $p \in [2, \infty]$, for small ε we may expand $V(\rho_\varepsilon)$ and integrate to get

$$
\int_{\mathbf{R}^2} V(\rho_\varepsilon^2)\, dx\, dy = 2\varepsilon \int_{\mathbf{R}^2} w^2 dx\, dy + \varepsilon^3 \left(2 - \frac{4}{3}F''(1)\right) \int_{\mathbf{R}^2} w^3 dx\, dy + \mathcal{O}(\varepsilon^5)
$$

$$
= \tfrac{3}{2} v_s^4 \mathscr{S}(w)\varepsilon - \tfrac{6}{\gamma}\left(v_s^2 - \tfrac{4}{3}F''(1)\right)\mathscr{S}(w)\varepsilon^3 + \mathcal{O}(\varepsilon^5).
$$

From the previous computations we find

$$
E(U_\varepsilon) - v_s Q(U_\varepsilon) = v_s^2 \mathscr{S}(w)\left(\frac{3}{2} - \frac{12 - 4F''(1)}{\gamma}\right)\varepsilon^3 + \mathcal{O}(\varepsilon^5). \tag{4.65}
$$

If $F''(1) \neq 3$, choose $\gamma \in \mathbf{R}$ such that $\frac{3}{2} - \frac{12-4F''(1)}{\gamma} < 0$ (take, for instance, $\gamma = 6 - 2F''(1)$). Let w be a ground state of (4.59) for this choice of γ. It follows from (4.65) that there is $\varepsilon_0 > 0$ such that $E(U_\varepsilon) - v_s Q(U_\varepsilon) < 0$ for any $\varepsilon \in (0, \varepsilon_0)$ (since $\mathscr{S}(w) > 0$). On the other hand, using (4.63) we infer that there is $\varepsilon_1 < \varepsilon_0$ such that the mapping $\varepsilon \longmapsto Q(U_\varepsilon)$ is a homeomorphism from $(0, \varepsilon_1)$ to an interval $(0, q_1)$. Since $E_{min}(Q(U_\varepsilon)) \le \overline{E}(U_\varepsilon) = E(U_\varepsilon) < v_s Q(U_\varepsilon)$, we see that $E_{min}(q) < v_s q$ for any $q \in (0, q_1)$. Then the concavity of E_{min} implies $E_{min}(q) < v_s q$ for any $q > 0$. $\square$

We pursue with some qualitative properties of E_{min} for large q. Theorem 4.17 (a) below implies that the speeds of traveling waves obtained from Theorem 4.9 tend to 0 as $q \longrightarrow \infty$.

Theorem 4.17

(a) If (A1) holds and $N \ge 2$, there is $C > 0$ such that $E_{min}(q) \le C q^{\frac{N-2}{N-1}} \ln q$ for large q.

(b) If $N \ge 2$ and (A1) and (A2) hold we have $\lim\limits_{q \to \infty} E_{min}(q) = \infty$. Moreover, if $N \ge 3$ there is $C > 0$ such that $E_{min}(q) \ge C q^{\frac{N-2}{N-1}}$.

Proof. (a) Using Lemma 4.4 p. 147 in [43] we see that there is a continuous mapping $R \longmapsto v_R$ from $[2, \infty)$ to $H^1(\mathbf{R}^N)$ and constants $C_i > 0$, $i = 1, 2, 3$, such that

$$
\int_{\mathbf{R}^N} |\nabla v_R|^2 dx \le C_1 R^{N-2} \ln R, \qquad \int_{\mathbf{R}^N} \|V(|1 + v_R|^2)\|\, dx \le C_2 R^{N-2}, \tag{4.66}
$$

$$
C_3 (R-2)^{N-1} \le Q(1 + v_R) \le C_3 R^{N-1}. \tag{4.67}
$$

Let $q_R = Q(1 + v_R)$. The set $\{q_R \mid R \ge 2\}$ is an interval of the form $[q_*, \infty)$. By (4.67) we have $C_3^{-\frac{1}{N-1}} q_R^{\frac{1}{N-1}} \le R \le 2 + C_3^{-\frac{1}{N-1}} q_R^{\frac{1}{N-1}}$. Then using (4.66) we get for R sufficiently large

$$
E_{min}(q_R) \le \overline{E}(1 + v_R) \le C_1 R^{N-2} \ln R + C_2 R^{N-2} \le C q_R^{\frac{N-2}{N-1}} \ln q_R.
$$

(b) As in the proof of Lemma 4.7 (ii), using (4.9) we get $E_{min}(q_2) \geq \left(\frac{q_2}{q_1}\right)^{\frac{N-2}{N-1}} E_{min}(q_1)$ for any $q_2 > q_1 > 0$. This is the second statement of (b), and it implies that $\lim_{q \to \infty} E_{min}(q) = \infty$ if $N \geq 3$.

Let $N = 2$. We argue by contradiction and we assume that $\lim_{q \to \infty} E_{min}(q)$ is finite. Using Theorem 4.9 for q sufficiently large, we may choose $\psi_q \in \mathcal{E}$ such that $Q(\psi_q) = q$ and $\overline{E}(\psi_q) = E_{min}(q)$. Consider a sequence $q_n \longrightarrow \infty$. From Lemma 4.8 it follows that $E_{GL}(\psi_{q_n})$ is bounded and stays away from 0. Passing to a subsequence we may assume that $E_{GL}(\psi_{q_n}) \longrightarrow \alpha_0 > 0$. Let $\Lambda_n(t)$ be the concentration function associated to $E_{GL}(\psi_{q_n})$ (as in (4.15)). Arguing as in the proof of Theorem 4.9 and passing to a subsequence (still denoted $(q_n)_{n \geq 1}$), we see that there exist a nondecreasing function $\Lambda : [0, \infty) \longrightarrow \mathbf{R}$, $\alpha \in [0, \alpha_0]$ and a sequence $t_n \longrightarrow \infty$ satisfying (4.16) and (4.17). Then we use Lemma 4.10 to infer that $\alpha > 0$.

If $\alpha \in (0, \alpha_0)$, proceeding as in the proof of Theorem 4.9 and using Lemma 3.3 for ψ_{q_n} we see that there exist functions $\psi_{n,1}, \psi_{n,2} \in \mathcal{E}$ such that (4.28)$-$(4.30) hold. Passing to a subsequence if necessary, we may assume that $\overline{E}(\psi_{n,i}) \longrightarrow m_i \geq 0$ as $n \longrightarrow \infty$. Since $\lim_{n \to \infty} E_{GL}(\psi_{n,i}) > 0$, it follows from Lemma 4.1 that $m_i > 0$, $i = 1, 2$. Using (4.29) we see that $m_1 + m_2 = \lim_{q \to \infty} E_{min}(q)$, hence $0 < m_i < \lim_{q \to \infty} E_{min}(q)$. Since $Q(\psi_{q_n}) = q_n \longrightarrow \infty$, from (4.30) it follows that at least one of the sequences $(Q(\psi_{n,i}))_{n \geq 1}$ contains a subsequence $(Q(\psi_{n_k,i}))_{k \geq 1}$ that tends to ∞. Then $\overline{E}(\psi_{n_k,i}) \geq E_{min}(Q(\psi_{n_k,i}))$ and passing to the limit as $k \longrightarrow \infty$ we find $m_i \geq \lim_{q \to \infty} E_{min}(q)$, a contradiction. Thus we cannot have $\alpha \in (0, \alpha_0)$.

We conclude that necessarily $\alpha = \alpha_0$. Proceeding again as in the proof of Theorem 4.9 we infer that there is a sequence $(x_n)_{n \geq 1} \subset \mathbf{R}^N$ such that $\tilde{\psi}_n = \psi_{q_n}(\cdot + x_n)$ satisfies (4.31). Then there is a subsequence $(\tilde{\psi}_{n_k})_{k \geq 1}$ and $\psi \in \mathcal{E}$ such that (4.32) holds. Using Lemma 4.12 we infer that $Q(\tilde{\psi}_{n_k}) \longrightarrow Q(\psi) \in \mathbf{R}$ and this is in contradiction with $Q(\tilde{\psi}_{n_k}) = q_{n_k} \longrightarrow \infty$. Thus necessarily $E_{min}(q) \longrightarrow \infty$ as $q \longrightarrow \infty$. $\square$

An alternative proof of the fact that $E_{min}(q) \longrightarrow \infty$ as $q \longrightarrow \infty$ is to show that for $\psi \in \mathcal{E}$ we may write $\langle i\psi_{x_1}, \psi \rangle = f + g$, where $g \in \mathcal{Y}$ and f is bounded in $L^1(\mathbf{R}^N)$ if $E_{GL}(\psi)$ is bounded, then to use Lemma 4.8 to infer that $Q(\psi)$ remains bounded if $\overline{E}(\psi)$ is bounded.

From Theorem 4.17 and Lemma 4.8 we obtain the following:

Corollary 4.18 *For all $M > 0$, the functional Q is bounded on the set $\{\psi \in \mathcal{E} \mid E_{GL}(\psi) \leq M\}$. If (A1) and (A2) hold, Q is also bounded on $\{\psi \in \mathcal{E} \mid \overline{E}(\psi) \leq M\}$.*

5 Minimizing the action at fixed kinetic energy

Although in many important physical applications the nonlinear potential V may achieve negative values (this happens, for instance, for the cubic-quintic NLS), there are no results in the literature that imply the existence of finite energy traveling waves for (1.1) in space dimension two for this kind of nonlinearity. We develop here a method that works if $N \geq 2$ and V takes negative values. The method used in [43] (minimization of E_c under a Pohozaev constraint) does not require any assumption on sign of the potential V, hence can be applied for the cubic-quintic NLS if $N \geq 3$, but does not work in space dimension two. Throughout this section we assume that (A1) and (A2) are satisfied.

We begin with a refinement of Lemma 4.4.

Lemma 5.1 *Assume that $|c| < v_s$ and let $\varepsilon \in (0, 1 - \frac{|c|}{v_s})$. There is $k > 0$ such that for any $\psi \in \mathcal{E}$ satisfying $\int_{\mathbf{R}^N} |\nabla u|^2 \, dx \leq k$ we have*

$$E(\psi) - \varepsilon E_{GL}(\psi) \geq |cQ(\psi)|.$$

Proof. Fix $\varepsilon_1 > 0$ such that $\varepsilon + \varepsilon_1 < 1 - \frac{|c|}{v_s}$. It follows from Lemma 4.1 that there is $k_1 > 0$ such that

$$(5.1) \qquad (1-\varepsilon_1)E_{GL}(\psi) \le E(\psi) \qquad \text{for any } \psi \in \mathcal{E} \text{ satisfying } \int_{\mathbf{R}^N} |\nabla\psi|^2\,dx \le k_1.$$

Let $\tilde{F}(s) = (1-\varphi^2(\sqrt{s}))\varphi(\sqrt{s})\varphi'(\sqrt{s})\frac{1}{\sqrt{s}}$. Then $\tilde{F}(s) = 1 - s$ for $s \in [0,4]$ and $\tilde{F}$ satisfies (A1) and (A2). Let $\tilde{V}(s) = \int_s^1 \tilde{F}(\tau)\,d\tau = \frac{1}{2}\left(\varphi^2(\sqrt{s}) - 1\right)^2$. Using Lemma 4.4 (ii) with $\tilde{F}$ and $\tilde{V}$ instead of F and V we infer that there is $k \in (0, \frac{k_1}{2})$ such that for any $\psi \in \mathcal{E}$ with $E_{GL}(\psi) \le 2k$ we have

$$(5.2) \qquad (1-\varepsilon-\varepsilon_1)E_{GL}(\psi) \ge |cQ(\psi)|.$$

Let $\psi \in \mathcal{E}$ be such that $\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx \le k$.

If $\frac{1}{2}\int_{\mathbf{R}^N}\left(\varphi^2(|\psi|)^2 - 1\right)^2\,dx \le k$ we have $E_{GL}(\psi) \le 2k$ and (5.2) holds. Then using (5.1) we obtain $E(\psi) - \varepsilon E_{GL}(\psi) \ge (1-\varepsilon-\varepsilon_1)E_{GL}(\psi) \ge |cQ(\psi)|$.

If $\frac{1}{2}\int_{\mathbf{R}^N}\left(\varphi^2(|\psi|)^2 - 1\right)^2\,dx > k$, let $\sigma = \left(\int_{\mathbf{R}^N}|\nabla\psi|^2\,dx\right)^{\frac{1}{2}}\left(\frac{1}{2}\int_{\mathbf{R}^N}\left(\varphi^2(|\psi|)^2 - 1\right)^2\,dx\right)^{-\frac{1}{2}}$. Then $\sigma \in (0,1)$ and

$$\frac{1}{2}\int_{\mathbf{R}^N}\left(\varphi^2(|\psi_{\sigma,\sigma}|)^2 - 1\right)^2\,dx = \int_{\mathbf{R}^N}|\nabla\psi_{\sigma,\sigma}|^2\,dx = \frac{1}{2}E_{GL}(\psi_{\sigma,\sigma}) = \sigma^{N-2}\int_{\mathbf{R}^N}|\nabla\psi|^2\,dx < k.$$

Using (5.1) and (5.2) we get $E(\psi) \ge (1-\varepsilon_1)E_{GL}(\psi)$ and $(1-\varepsilon-\varepsilon_1)E_{GL}(\psi_{\sigma,\sigma}) \ge |cQ(\psi_{\sigma,\sigma})|$. Then we have

$$E(\psi) - \varepsilon E_{GL}(\psi) - |cQ(\psi)| \ge (1-\varepsilon-\varepsilon_1)E_{GL}(\psi) - |cQ(\psi)|$$

$$\ge (1-\varepsilon-\varepsilon_1)\left(\frac{1}{\sigma^{N-2}}\int_{\mathbf{R}^N}|\nabla\psi_{\sigma,\sigma}|^2\,dx + \frac{1}{2\sigma^N}\int_{\mathbf{R}^N}\left(\varphi^2(|\psi_{\sigma,\sigma}|)^2 - 1\right)^2\,dx\right) - \frac{1}{\sigma^{N-1}}|cQ(\psi_{\sigma,\sigma})|$$

$$\ge \frac{1-\varepsilon-\varepsilon_1}{2}\left(\frac{1}{\sigma^{N-2}} + \frac{1}{\sigma^N}\right)E_{GL}(\psi_{\sigma,\sigma}) - \frac{1-\varepsilon-\varepsilon_1}{\sigma^{N-1}}E_{GL}(\psi_{\sigma,\sigma}) \ge 0.$$

□

Let $I(\psi) = -Q(\psi) + \int_{\mathbf{R}^N} V(|\psi|^2)\,dx = E(\psi) - Q(\psi) - \int_{\mathbf{R}^N}|\nabla\psi|^2\,dx$.
We will minimize $I(\psi)$ under the constraint $\|\nabla\psi\|_{L^2(\mathbf{R}^N)} = constant$. For any $k > 0$ we define

$$I_{min}(k) = \inf\left\{I(\psi) \;\Big|\; \psi \in \mathcal{E}, \int_{\mathbf{R}^N}|\nabla\psi|^2\,dx = k\right\}.$$

The next Lemmas establish the basic properties of the function I_{min}.

Lemma 5.2 *(i) For any $k > 0$ we have $I_{min}(k) \le -\frac{1}{v_s^2}k$.*

(ii) For any $\delta > 0$ there is $k(\delta) > 0$ such that $I_{min}(k) \ge -\frac{1+\delta}{v_s^2}k$ for any $k \in (0, k(\delta))$.

Proof. i) Let $N \ge 3$. Let $q = 2kv_s^{N-3}$. In the proof of Lemma 4.5 we have constructed a sequence $(\psi_n)_{n\ge1} \subset \mathcal{E}$ such that

$$Q(\psi_n) = q, \qquad \int_{\mathbf{R}^N}|\nabla\psi_n|^2\,dx \longrightarrow \frac{1}{2}v_s q = kv_s^{N-2}, \qquad \int_{\mathbf{R}^N} V(|\psi_n|^2)\,dx \longrightarrow \frac{1}{2}v_s q$$

and ψ_n is constant outside a large ball. Let $\sigma_n = k^{\frac{1}{N-2}}\left(\int_{\mathbf{R}^N}|\nabla\psi_n|^2\,dx\right)^{-\frac{1}{N-2}}$. Then $\sigma_n \longrightarrow \frac{1}{v_s}$ as $n \longrightarrow \infty$. We get

$$\int_{\mathbf{R}^N}|\nabla((\psi_n)_{\sigma_n,\sigma_n})|^2\,dx = \sigma_n^{N-2}\int_{\mathbf{R}^N}|\nabla\psi_n|^2\,dx = k,$$

$$Q((\psi_n)_{\sigma_n,\sigma_n}) = \sigma_n^{N-1} Q(\psi_n) \longrightarrow \frac{q}{v_s^{N-1}} = \frac{2k}{v_s^2},$$

$$\int_{\mathbf{R}^N} V(|(\psi_n)_{\sigma_n,\sigma_n}|^2)\,dx = \sigma_n^N \int_{\mathbf{R}^N} V(|\psi_n|^2)\,dx \longrightarrow \frac{1}{v_s^N}\cdot\frac{v_s q}{2} = \frac{k}{v_s^2}.$$

We have $I_{min}(k) \le I((\psi_n)_{\sigma_n,\sigma_n})$ for each n and passing to the limit as $n \longrightarrow \infty$ we obtain $I_{min}(k) \le -\frac{1}{v_s^2}k$.

If $N=2$, let $q = \frac{2k}{v_s}$, choose ψ_n as in the proof of Lemma 4.5 such that

$$\int_{\mathbf{R}^2} |\nabla\psi_n|^2\,dx = k, \qquad Q(\psi_n) \longrightarrow q = \frac{2k}{v_s} \qquad \text{and} \qquad \int_{\mathbf{R}^2} V(|\psi_n|^2)\,dx \longrightarrow k.$$

Let $\sigma = \frac{1}{v_s}$. Then $\int_{\mathbf{R}^2} |\nabla((\psi_n)_{\sigma,\sigma})|^2\,dx = k$, $Q((\psi_n)_{\sigma,\sigma}) = \sigma Q(\psi_n) \longrightarrow \frac{2k}{v_s^2}$ and $\int_{\mathbf{R}^2} V(|(\psi_n)_{\sigma,\sigma}|^2)\,dx = \sigma^2 \int_{\mathbf{R}^2} V(|\psi_n|^2)\,dx \longrightarrow \frac{k}{v_s^2}$, hence $I((\psi_n)_{\sigma,\sigma}) \longrightarrow -\frac{k}{v_s^2}$.

(ii) Fix $\delta > 0$ and let $c = \frac{v_s}{\sqrt{1+\delta}}$. Lemma 5.1 implies that there is $k>0$ such that for any $\psi \in \mathcal{E}$ with $\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx \le k$ we have

$$(5.3) \qquad \int_{\mathbf{R}^N} |\nabla\psi|^2\,dx - cQ(\psi) + \int_{\mathbf{R}^2} V(|\psi|^2)\,dx \ge 0.$$

Let $\psi \in \mathcal{E}$ be such that $\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx \le \frac{k}{c^{N-2}}$. Then $\int_{\mathbf{R}^N} |\nabla\psi_{c,c}|^2\,dx = c^{N-2}\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx \le k$, hence $\psi_{c,c}$ satisfies (5.3), that is $c^{N-2}\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx + c^N I(\psi) \ge 0$ or equivalently

$$I(\psi) \ge -\frac{1}{c^2}\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx = -\frac{1+\delta}{v_s^2}\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx.$$

Hence (ii) holds with $k(\delta) = \frac{k}{c^{N-2}}$. $\square$

We give now global properties of I_{min}.

Lemma 5.3 *The function I_{min} has the following properties:*

(i) I_{min} is concave, decreasing on $[0,\infty)$ and $\lim\limits_{k\to\infty} \frac{I_{min}(k)}{k} = -\infty$.

(ii) I_{min} is subadditive, that is $I_{min}(k_1+k_2) \le I_{min}(k_1) + I_{min}(k_2)$ for any $k_1, k_2 \ge 0$.

(iii) If either $N \ge 3$ or ($N=2$ and $V \ge 0$ on $[0,\infty)$), we have $I_{min}(k) > -\infty$ for any $k>0$.

(iv) If ($N=2$ and $\inf V < 0$), then $I_{min}(k) = -\infty$ for all sufficiently large k.

(v) Assume that $N=2$, (A4) holds and $F''(1) \ne 3$. Then $I_{min}(k) < -\frac{1}{v_s^2}k$ for any $k>0$.

Proof. i) We prove that for any $k>0$,

$$(5.4) \qquad I_{min}(k) \ge \limsup_{h\downarrow k} I_{min}(h).$$

Fix $\psi \in \mathcal{E}$ such that $\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx = k$. At least one of the mappings $t \longmapsto \int_{\mathbf{R}^N} |\nabla\psi_{t,t}|^2\,dx$, $t \longmapsto \int_{\mathbf{R}^N} |\nabla\psi_{1,t}|^2\,dx$ or $t \longmapsto \int_{\mathbf{R}^N} |\nabla\psi_{t,1}|^2\,dx$ is (strictly) increasing on $[1,\infty)$. Let ψ^t be either $\psi_{t,t}$ or $\psi_{1,t}$ or $\psi_{t,1}$, in such a way that $t \longmapsto \int_{\mathbf{R}^N} |\nabla\psi^t|^2\,dx$ is continuous and increasing on $[0,\infty)$. It is easy to see that $I(\psi^t) \longrightarrow I(\psi)$ as $t \longrightarrow 1$. Let $(k_n)_{n\ge1}$ be a sequence satisfying $k_n \downarrow k$. There is a sequence $t_n \downarrow 1$ such that $\int_{\mathbf{R}^N} |\nabla\psi^{t_n}|^2\,dx = k_n$. For each n we have $I_{min}(k_n) \le I(\psi^{t_n})$ and passing to the limit as $n \longrightarrow \infty$ we find $\limsup\limits_{n\to\infty} I_{min}(k_n) \le I(\psi)$. Since this is true for any sequence $k_n \downarrow k$ and any $\psi \in \mathcal{E}$ satisfying $\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx = k$, (5.4) follows.

Proceeding exactly as in the proof of Lemma 4.7 (see the proof of (4.12) there) we find

$$I_{min}\left(\frac{k_1+k_2}{2}\right) \geq \frac{1}{2}(I_{min}(k_1)+I_{min}(k_2)) \qquad \text{for any } k_1,\, k_2>0. \tag{5.5}$$

Let $0 \leq k_1 < k_2$. Using (5.5) and a straightforward induction we find

$$I_{min}(\alpha k_1+(1-\alpha)k_2) \geq \alpha I_{min}(k_1)+(1-\alpha)I_{min}(k_2) \qquad \text{for any } \alpha\in[0,1]\cap\mathbf{Q}. \tag{5.6}$$

Let $\alpha \in (0,1)$. Consider a sequence $(\alpha_n)_{n\geq 1}\subset[0,1]\cap\mathbf{Q}$ such that $\alpha_n \uparrow \alpha$. Using (5.4) and (5.6) we get

$$\begin{aligned}&I_{min}(\alpha k_1+(1-\alpha)k_2) \geq \limsup_{n\to\infty} I_{min}(\alpha_n k_1+(1-\alpha_n)k_2)\\ &\geq \limsup_{n\to\infty}\left(\alpha_n I_{min}(k_1)+(1-\alpha_n)I_{min}(k_2)\right)=\alpha I_{min}(k_1)+(1-\alpha)I_{min}(k_2).\end{aligned}$$

Thus I_{min} is concave on $[0,\infty)$. Since $I_{min}(0)=0$, by Lemma 5.2 I_{min} is continuous at 0 and negative on an interval $(0,\delta)$ and we infer that I_{min} is negative and decreasing on $(0,\infty)$.

The concavity of I_{min} implies that the function $k \longmapsto \frac{I_{min}(k)}{k}$ is nonincreasing on $(0,\infty)$. Using Lemma 4.4 in [43] we find a sequence $(\psi_n)_{n\geq 3}\subset\mathcal{E}$ such that

$$k_n := \int_{\mathbf{R}^N}|\nabla\psi_n|^2\,dx \leq C_1 n^{N-2}\ln n, \quad \left|\int_{\mathbf{R}^N}V(|\psi_n|^2)\,dx\right| \leq C_2 n^{N-2} \ \text{ and } \ Q(\psi_n)\geq C_3 n^{N-1},$$

where $C_1,\, C_2,\, C_3>0$ do not depend on n. Then $\lim\limits_{k\to\infty}\frac{I_{min}(k)}{k} \leq \lim\limits_{n\to\infty}\frac{I(\psi_n)}{k_n}=-\infty$.

(ii) By concavity we have $I_{min}(k_i) \geq \frac{k_i}{k_1+k_2} I_{min}(k_1+k_2)$, $i=1,2$, and the subadditivity follows.

(iii) Consider first the case $N\geq 3$. Fix $k>0$. Argue by contradiction and assume that there is a sequence $(\psi_n)_{n\geq 1}\subset\mathcal{E}$ such that $\int_{\mathbf{R}^N}|\nabla\psi_n|^2\,dx=k$ and

$$I(\psi_n) = -Q(\psi_n)+\int_{\mathbf{R}^N}V(|\psi_n|^2)\,dx \longrightarrow -\infty \qquad \text{as } n\longrightarrow\infty. \tag{5.7}$$

Let $c=\frac{v_s}{2}$. By Lemma 5.1 there exists $k_2>0$ such that $\frac{k_2}{k}<\left(\frac{v_s}{2}\right)^{N-2}$ and (5.3) holds for any $\psi\in\mathcal{E}$ with $\int_{\mathbf{R}^N}|\nabla\psi|^2\,dx\leq k_2$. Let $\sigma = k_2^{\frac{1}{N-2}}k^{-\frac{1}{N-2}} < \frac{v_s}{2}$. Then $\int_{\mathbf{R}^N}|\nabla((\psi_n)_{\sigma,\sigma}|^2\,dx = k_2$, hence $(\psi_n)_{\sigma,\sigma}$ satisfies (5.3), that is

$$\int_{\mathbf{R}^N}|\nabla\psi_n|^2\,dx - \sigma\frac{v_s}{2}Q(\psi_n)+\sigma^2\int_{\mathbf{R}^N}V(|\psi_n|^2)\,dx \geq 0. \tag{5.8}$$

From (5.7) and (5.8) we get

$$-\int_{\mathbf{R}^N}|\nabla\psi_n|^2\,dx+\left(\sigma\frac{v_s}{2}-\sigma^2\right)\int_{\mathbf{R}^N}V(|\psi_n|^2)\,dx \longrightarrow -\infty,$$

which implies $\int_{\mathbf{R}^N}V(|\psi_n|^2)\,dx \longrightarrow -\infty$ as $n\longrightarrow\infty$. Since $\int_{\mathbf{R}^N}|\nabla\psi_n|^2\,dx=k$, this contradicts the first inequality in (4.1).

Next assume that $N=2$ and $V\geq 0$ on $[0,\infty)$. Fix $k>0$. By Corollary 4.18 there is $q_k>0$ such that $|Q(\psi)|\leq q_k$ for any $\psi\in\mathcal{E}$ satisfying $E(\psi)\leq k+1$. Let $\psi\in\mathcal{E}$ be such that $\int_{\mathbf{R}^2}|\nabla\psi|^2\,dx=k$. If $\int_{\mathbf{R}^2}V(|\psi|^2)\,dx=0$ we infer that $|\psi|=1$ a.e. on $\mathbf{R}^2$ and then (2.7) implies $Q(\psi)=0$, hence $I(\psi)=0$. If $\int_{\mathbf{R}^2}V(|\psi|^2)\,dx>0$ let $\sigma=\left(\int_{\mathbf{R}^2}V(|\psi|^2)\,dx\right)^{-\frac{1}{2}}$ and $\tilde\psi=\psi_{\sigma,\sigma}$, so that $\int_{\mathbf{R}^2}V(|\tilde\psi|^2)\,dx=1$ and $\int_{\mathbf{R}^2}|\nabla\tilde\psi|^2\,dx=k$. We infer that $|Q(\tilde\psi)|\leq q_k$. Since $\psi=\tilde\psi_{\frac{1}{\sigma},\frac{1}{\sigma}}$

we have by scaling $I(\psi) = \sigma^{-2}\int_{\mathbf{R}^2} V(|\tilde{\psi}|^2)\,dx - \sigma^{-1}Q(\tilde{\psi}) \geq \sigma^{-2} - \sigma^{-1}q_k \geq -\frac{q_k^2}{4}$. We conclude that $I_{min}(k) \geq -\frac{q_k^2}{4} > -\infty$.

iv) If V achieves negative values, it is easy to see that there exists $\psi_1 \in \mathcal{E}$ such that $\int_{\mathbf{R}^2} V(|\psi_1|^2)\,dx < 0$. Let $k_1 = \int_{\mathbf{R}^2} |\nabla\psi_1|^2\,dx$. Then, for any $t > 0$, $\int_{\mathbf{R}^2} |\nabla(\psi_1)_{t,t}|^2\,dx = \int_{\mathbf{R}^2} |\nabla\psi_1|^2\,dx = k_1$ because $N = 2$, thus

$$I_{min}(k_1) \leq I((\psi_1)_{t,t}) = -tQ(\psi_1) + t^2\int_{\mathbf{R}^2} V(|\psi_1|^2)\,dx \longrightarrow -\infty$$

as $t \to \infty$. By concavity we have $I_{min}(k) = -\infty$ for any $k \geq k_1$.

v) The proof relies on the comparison maps constructed in the proof of Theorem 4.15 from the (KP-I) ground state. Notice first that if $\psi \in \mathcal{E}$ is such that $\int_{\mathbf{R}^2} |\nabla\psi|^2\,dx = k$, $\int_{\mathbf{R}^2} V(|\psi|^2)\,dx > 0$ and $Q(\psi) > 0$, then the function $t \longmapsto I(\psi_{t,t}) = t^2\int_{\mathbf{R}^2} V(|\psi|^2)\,dx - tQ(\psi)$ achieves its minimum at $t_0 = \frac{1}{2}Q(\psi)\left(\int_{\mathbf{R}^2} V(|\psi|^2)\,dx\right)^{-1}$. Since $\int_{\mathbf{R}^2} |\nabla\psi|^2\,dx = \int_{\mathbf{R}^2} |\nabla\psi_{t,t}|^2\,dx = k$ in dimension $N = 2$, it follows that

$$I_{min}(k) \leq \inf_{t>0} I(\psi_{t,t}) = I(\psi_{t_0,t_0}) = -\frac{Q^2(\psi)}{4\int_{\mathbf{R}^2} V(|\psi|^2)\,dx}. \tag{5.9}$$

Fix $\gamma \neq 0$ (to be chosen later), and let w be a ground state for (4.59). Then, for ε small enough, we have seen in the proof of Theorem 4.15 how to construct from w a comparison map $U_\varepsilon \in \mathcal{E}$ satisfying

$$Q(U_\varepsilon) = 3v_s^3\mathscr{S}(w)\varepsilon - \frac{6}{\gamma}v_s\mathscr{S}(w)\varepsilon^3,$$

$$\int_{\mathbf{R}^2} V(|U_\varepsilon|^2)\ dx = \frac{3}{2}v_s^4\mathscr{S}(w)\varepsilon - \frac{6}{\gamma}\left(v_s^2 - \frac{4}{3}F''(1)\right)\mathscr{S}(w)\varepsilon^3 + \mathcal{O}(\varepsilon^5),$$

$$\int_{\mathbf{R}^2} |\nabla U_\varepsilon|^2\ dx = \frac{3}{2}v_s^4\mathscr{S}(w)\varepsilon + v_s^2\mathscr{S}(w)\left(\frac{3}{2} - \frac{12}{\gamma}\right)\varepsilon^3 + \mathcal{O}(\varepsilon^5).$$

Let $k_\varepsilon = \int_{\mathbf{R}^2} |\nabla U_\varepsilon|^2\ dx$. Since $Q(U_\varepsilon) > 0$ and $\int_{\mathbf{R}^2} V(|U_\varepsilon|^2)\ dx > 0$ for ε small, we infer from (5.9) that

$$\begin{aligned} I_{min}(k_\varepsilon) \leq\ & -\frac{Q^2(U_\varepsilon)}{4\int_{\mathbf{R}^2} V(|U_\varepsilon|^2)\,dx} = -\frac{3}{2}v_s^2\mathscr{S}(w)\varepsilon + \frac{4F''(1)}{\gamma}\mathscr{S}(w)\varepsilon^3 + \mathcal{O}(\varepsilon^5) \\ =\ & -\frac{1}{v_s^2}\left[\frac{3}{2}v_s^4\mathscr{S}(w)\varepsilon - v_s^2\frac{4F''(1)}{\gamma}\mathscr{S}(w)\varepsilon^3 + \mathcal{O}(\varepsilon^5)\right]. \end{aligned}$$

Therefore, we have

$$I_{min}(k_\varepsilon) < -\frac{k_\varepsilon}{v_s^2}$$

for all ε sufficiently small provided that $-\frac{4}{\gamma}F''(1) \geq \frac{3}{2} - \frac{12}{\gamma}$, that is $\frac{4(3-F''(1))}{\gamma} > \frac{3}{2}$ (take, for instance, $\gamma = 3 - F''(1)$). $\square$

Let

$$k_0 = \inf\left\{k \geq 0 \;\middle|\; I_{min}(k) < -\frac{1}{v_s^2}k\right\} \qquad \text{and} \qquad k_\infty = \inf\{k > 0 \mid I_{min}(k) = -\infty\}. \tag{5.10}$$

By Lemmas 5.2 and 5.3 (i) we have $0 \le k_0 < \infty$ and $0 < k_\infty \le \infty$. It is clear that $k_0 \le k_\infty$. If either $N \ge 3$ or $N = 2$ and $V \ge 0$ on $[0, \infty)$ we have $k_\infty = \infty$, while if $N = 2$ and (A4) holds with $F''(1) \ne 3$, we have $k_0 = 0$; obviously, in all these cases we have $k_0 < k_\infty$. The next Lemma gives further information in the case when $N = 2$ and V achieves negative values. It brings into light the relationship between k_∞ and the Dirichlet energy of the stationary solutions of (1.1) with minimal energy, the so-called ground states or bubbles.

Lemma 5.4 *Assume that* $N = 2$, *(A1), (A2) are satisfied and* $\inf V < 0$. *Let*

$$T = \inf \left\{ \int_{\mathbf{R}^2} |\nabla \psi|^2 \, dx \;\Big|\; \psi \in \mathcal{E},\ |\psi| \text{ is not constant and } \int_{\mathbf{R}^2} V(|\psi|^2)\, dx \le 0 \right\}.$$

Then:

(i) We have $T > 0$ *and the infimum is achieved for some* $\psi_0 \in \mathcal{E}$. *Moreover, any such* ψ_0 *satisfies the equation* $\Delta \psi_0 + \sigma^2 F(|\psi_0|^2)\psi_0 = 0$ *in* $\mathcal{D}'(\mathbf{R}^2)$ *for some* $\sigma > 0$, $\int_{\mathbf{R}^2} V(|\psi_0|^2)\, dx = 0$, ψ_0 *belongs to* $C^{1,\alpha}(\mathbf{R}^2)$ *for any* $\alpha \in (0,1)$ *and, after a translation,* ψ_0 *is radially symmetric.*

(ii) For any $k < T$ *and any* $M > 0$, E_{GL} *is bounded on the set*

$$\mathcal{E}_{k,M} := \left\{ \psi \in \mathcal{E} \;\Big|\; \int_{\mathbf{R}^2} |\nabla \psi|^2\, dx \le k,\ \int_{\mathbf{R}^2} V(|\psi|^2)\, dx \le M \right\}.$$

(iii) We have $k_\infty = T$.

Proof. (i) It follows from Corollary 4.2 that $T > 0$. The proof of the existence and regularity of minimizers is rather classical and is similar to the proof of Theorem 3.1 p. 106 in [12], so we omit it. Notice that any minimizer of the considered problem is also a minimizer of $\int_{\mathbf{R}^2} V(|\psi|^2)\, dx$ under the constraint $\int_{\mathbf{R}^2} |\nabla \psi|^2\, dx = T$ and then the radial symmetry follows from Theorem 2 p. 314 in [42].

(ii) Fix $\beta \in (0, 1]$ such that

$$V(s^2) \ge \frac{1}{4}(s^2 - 1)^2 \qquad \text{for any } s \in ((1-\beta)^2, (1+\beta)^2). \tag{5.11}$$

It suffices to prove that for any sequence $(\psi_n)_{n\ge1} \subset \mathcal{E}_{k,M}$, $E_{GL}(\psi_n)$ is bounded. Let $(\psi_n)_{n\ge1} \subset \mathcal{E}_{k,M}$. Let $K_n = \{x \in \mathbf{R}^2 \mid \big| |\psi_n(x)| - 1 \big| \ge \frac{\beta}{2}\}$. We claim that it suffices to prove that $\mathcal{L}^2(K_n)$ is bounded.

Indeed, assume $\mathcal{L}^2(K_n)$ bounded. Let $\tilde{\psi}_n = \left(\big| |\psi_n| - 1 \big| - \frac{\beta}{2} \right)_+$. Then $\tilde{\psi}_n \in L^1_{loc}(\mathbf{R}^2)$, $|\nabla \tilde{\psi}_n| \le |\nabla \psi_n|$ a.e. on $\mathbf{R}^2$ and by (2.2) we have

$$\int_{\mathbf{R}^2} |\tilde{\psi}_n|^{2p_0+2}\, dx \le C_{2p_0+2} \|\nabla \tilde{\psi}_n\|^{2p_0+2}_{L^2(\mathbf{R}^2)} \mathcal{L}^2(K_n) \le C_{2p_0+2} \|\nabla \psi_n\|^{2p_0+2}_{L^2(\mathbf{R}^2)} \mathcal{L}^2(K_n).$$

By (A1) and (A2) there is $C_0 > 0$ such that $|V(s^2)| \le C_0 \left(|s-1| - \frac{\beta}{2} \right)^{2p_0+2}$ for any s satisfying $|s - 1| \ge \beta$. Hence

$$\int_{\mathbf{R}^2 \setminus \{1-\beta \le |\psi_n| \le 1+\beta\}} |V(|\psi_n|^2)|\, dx \le C_0 \int_{\mathbf{R}^2} |\tilde{\psi}_n|^{2p_0+2}\, dx \le C_0 C_{2p_0+2} \|\nabla \psi_n\|^{2p_0+2}_{L^2(\mathbf{R}^2)} \mathcal{L}^2(K_n)$$

and the last quantity is bounded. Since $\int_{\mathbf{R}^2} V(|\psi_n|^2)\, dx$ is bounded, we infer that $\int_{\{1-\beta \le |\psi_n| \le 1+\beta\}} V(|\psi_n|^2)\, dx$ is bounded, and by (5.11), $\int_{\{1-\beta \le |\psi_n| \le 1+\beta\}} \left(\varphi^2(|\psi_n|) - 1 \right)^2 dx$ is bounded. On the other hand, $\int_{\mathbf{R}^2\setminus\{1-\beta\le|\psi_n|\le 1+\beta\}}\left(\varphi^2(|\psi_n|)-1\right)^2 dx \le \int_{K_n}\left(\varphi^2(|\psi_n|)-1\right)^2 dx \le 64\mathcal{L}^2(K_n)$ and the conclusion follows.

It remains to prove the boundedness of $\mathcal{L}^2(K_n)$. Let

$$\psi_n^+ = \begin{cases} |\psi_n| & \text{if } |\psi_n|\ge 1 \\ 1 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \psi_n^- = \begin{cases} |\psi_n| & \text{if } |\psi_n|\le 1 \\ 1 & \text{if } |\psi_n|\ge 1. \end{cases}$$

It is clear that $\psi_n^+,\psi_n^-\in\mathcal{E}$, $\int_{\mathbf{R}^2}|\nabla\psi_n^+|^2+|\nabla\psi_n^-|^2\,dx = \int_{\mathbf{R}^2}|\nabla|\psi_n||^2\,dx\le k$ and $\int_{\mathbf{R}^2}V(|\psi_n^+|^2)+V(|\psi_n^-|^2)\,dx = \int_{\mathbf{R}^2}V(|\psi_n|^2)\,dx$. If $\int_{\mathbf{R}^2}V(|\psi_n^+|^2)\,dx<0$, by (i) we have $\int_{\mathbf{R}^2}|\nabla\psi_n^+|^2\,dx\ge T>k$, a contradiction. Thus necessarily $\int_{\mathbf{R}^2}V(|\psi_n^+|^2)\,dx\ge 0$ and similarly $\int_{\mathbf{R}^2}V(|\psi_n^-|^2)\,dx\ge 0$, hence $\int_{\mathbf{R}^2}V(|\psi_n^\pm|^2)\,dx\in[0,M]$.

Let $K_n^+ = \{x\in\mathbf{R}^2 \mid |\psi_n(x)|\ge 1+\frac{\beta}{2}\}$, $K_n^- = \{x\in\mathbf{R}^2 \mid |\psi_n(x)|\le 1-\frac{\beta}{2}\}$. Let $w_n^+ = \phi_n^+(|x|)$ and $w_n^- = \phi_n^-(|x|)$ be the symmetric decreasing rearrangements of $(|\psi_n|-1)_+ = \psi_n^+-1$ and of $(|\psi_n|-1)_- = 1-\psi_n^-$, respectively. As in the proof of Lemma 4.8 we have $\phi_n^\pm\in H^1_{loc}((0,\infty))$. Let

$$t_n = \inf\{t\ge 0 \mid \phi_n^+(t)<\tfrac{\beta}{2}\} \qquad \text{and} \qquad s_n = \inf\{t\ge 0 \mid \phi_n^-(t)<\tfrac{\beta}{2}\}.$$

Then $\mathcal{L}^2(K_n^+) = \mathcal{L}^2(\{(|\psi_n|-1)_+\ge\frac{\beta}{2}\}) = \mathcal{L}^2(\{w_n^+\ge\frac{\beta}{2}\}) = \mathcal{L}^2(\overline{B}(0,t_n)) = \pi t_n^2$ and similarly $\mathcal{L}^2(K_n^-) = \pi s_n^2$, so that $\mathcal{L}^2(K_n) = \pi(t_n^2+s_n^2)$.

Assume that there is a subsequence $t_{n_j}\longrightarrow\infty$. Let $\tilde{w}_j = (w_{n_j}^+)_{\frac{1}{t_{n_j}},\frac{1}{t_{n_j}}} = \phi_{n_j}^+(t_{n_j}|\cdot|)$, so that $\tilde{w}_j\ge\frac{\beta}{2}$ on $\overline{B}(0,1)$ and $0\le\tilde{w}_j<\frac{\beta}{2}$ on $\mathbf{R}^2\setminus\overline{B}(0,1)$. Then $\int_{\mathbf{R}^2}|\nabla\tilde{w}_j|^2\,dx = \int_{\mathbf{R}^2}|\nabla w_{n_j}|^2\,dx\le k$ and using (2.2) we see that $(\tilde{w}_j-\frac{\beta}{2})_+$ is uniformly bounded in $L^p(B(0,1))$ for any $p<\infty$, and consequently $(\tilde{w}_j)_{j\ge1}$ is bounded in $L^p(B(0,R))$ for any $p<\infty$ and any $R\in(0,\infty)$. Then there is a subsequence of $(\tilde{w}_j)_{j\ge1}$, still denoted $(\tilde{w}_j)_{j\ge1}$, and there is $\tilde{w}\in H^1_{loc}(\mathbf{R}^2)$ such that $\nabla\tilde{w}\in L^2(\mathbf{R}^2)$ and $(\tilde{w}_j)_{j\ge1}$, $\tilde{w}$ satisfy (4.32). It is easy to see that $1+\tilde{w}\in\mathcal{E}$ and $\tilde{w}\ge\frac{\beta}{2}$ on $\overline{B}(0,1)$. By weak convergence we have

$$\tilde{k} := \int_{\mathbf{R}^2}|\nabla\tilde{w}|^2\,dx \le \liminf_{j\to\infty}\int_{\mathbf{R}^2}|\nabla\tilde{w}_j|^2\,dx\le k.$$

Using (A2), the convergence $\tilde{w}_j\longrightarrow\tilde{w}$ in $L^{2p_0+2}(B(0,1))$ and Theorem A2 p. 133 in [46] we get $\int_{B(0,1)}V((1+\tilde{w}_j)^2)\,dx\longrightarrow\int_{B(0,1)}V((1+\tilde{w})^2)\,dx$. Since $\tilde{w}_j\in[0,\frac{\beta}{2}]$ on $\mathbf{R}^2\setminus B(0,1)$ and $V(s^2)\ge 0$ for $s\in[1,1+\frac{\beta}{2}]$, using Fatou's Lemma we obtain $\int_{\mathbf{R}^2\setminus B(0,1)}V((1+\tilde{w})^2)\,dx\le \liminf\limits_{j\to\infty}\int_{\mathbf{R}^2\setminus B(0,1)}V((1+\tilde{w}_j)^2)\,dx$. Therefore

$$\begin{aligned}\int_{\mathbf{R}^2}V((1+\tilde{w})^2)\,dx &\le \liminf_{j\to\infty}\int_{\mathbf{R}^2}V((1+\tilde{w}_j)^2)\,dx = \liminf_{j\to\infty}\frac{1}{t_{n_j}^2}\int_{\mathbf{R}^2}V((1+w_{n_j}^+)^2)\,dx \\ &= \liminf_{j\to\infty}\frac{1}{t_{n_j}^2}\int_{\mathbf{R}^2}V((\psi_{n_j}^+)^2)\,dx\le 0\end{aligned}$$

because $\int_{\mathbf{R}^2}V((\psi_{n_j}^+)^2)\,dx\le M$ and $t_{n_j}\to+\infty$ by our assumption. Since $1+\tilde{w}\ge 1+\frac{\beta}{2}$ on $B(0,1)$ we infer that $\int_{\mathbf{R}^2}|\nabla\tilde{w}|^2\,dx\ge T>k$, a contradiction.

So far we have proved that $(t_n)_{n\ge1}$ is bounded. Similarly $(s_n)_{n\ge1}$ is bounded, thus $(\mathcal{L}^2(K_n))_{n\ge1}$ is bounded and the proof of (ii) is complete.

(iii) Consider a radial function $\psi_0\in\mathcal{E}$ such that $|\psi_0|$ is not constant, $\int_{\mathbf{R}^2}V(|\psi_0|^2)\,dx=0$ and $\int_{\mathbf{R}^2}|\nabla\psi_0|^2\,dx = T$. Since $F(|\psi_0|^2)\psi_0$ does not vanish a.e. on $\mathbf{R}^2$, there exists a radial function $\phi\in C_c^\infty(\mathbf{R}^2)$ such that $\int_{\mathbf{R}^2}\langle F(|\psi_0|^2)\psi_0,\phi\rangle\,dx>0$. It follows that $\frac{d}{dt}_{|t=0}\int_{\mathbf{R}^2}V(|\psi_0+t\phi|^2)\,dx = -2\int_{\mathbf{R}^2}\langle F(|\psi_0|^2\psi_0,\phi\rangle\,dx<0$, consequently there is $\varepsilon>0$ such that $\int_{\mathbf{R}^2}V(|\psi_0+$

$t\phi|^2)\, dx < \int_{\mathbf{R}^2} V(|\psi_0|^2)\, dx = 0$ for any $t \in (0, \varepsilon)$. Denote $k(t) = \int_{\mathbf{R}^2} |\nabla(\psi_0 + t\phi)|^2\, dx$. It follows from the proof of Lemma 5.3 (iv) that $I_{min}(k(t)) = -\infty$ for any $t \in (0, \varepsilon)$, thus $k_\infty \le k(t)$ for any $t \in (0, \varepsilon)$. Since $k(t) \longrightarrow T$ as $t \longrightarrow 0$, we infer that $k_\infty \le T$.

Let $k < T$. Consider $\psi \in \mathcal{E}$ such that $\int_{\mathbf{R}^2} |\nabla\psi|^2\, dx = k$. If $|\psi| = 1$ a.e. we have $V(|\psi|^2) = 0$ a.e. and $Q(\psi) = 0$ by (2.7), hence $I(\psi) = 0$. If $|\psi|$ is not constant, then we have necessarily $\int_{\mathbf{R}^2} V(|\psi|^2)\, dx > 0$. If $Q(\psi) \le 0$, it is obvious that $I(\psi) > 0$. If $Q(\psi) > 0$ we have $\inf\limits_{t>0} I(\psi_{t,t}) = -\frac{1}{4} Q^2(\psi) \left(\int_{\mathbf{R}^2} V(|\psi|^2)\, dx\right)^{-1}$ and the infimum is achieved for $t_{min} = \frac{1}{2} Q(\psi) \left(\int_{\mathbf{R}^2} V(|\psi|^2)\, dx\right)^{-1}$. There exists $t_1 > 0$ such that $\int_{\mathbf{R}^2} V(|\psi_{t_1,t_1}|^2)\, dx = 1$. Then $\int_{\mathbf{R}^2} |\nabla\psi_{t_1,t_1}|^2\, dx = k$ and

$$I(\psi) \ge \inf_{t>0} I(\psi_{t,t}) = -\frac{Q^2(\psi)}{4\int_{\mathbf{R}^2} V(|\psi|^2)\, dx} = -\frac{Q^2(\psi_{t_1,t_1})}{4\int_{\mathbf{R}^2} V(|\psi_{t_1,t_1}|^2)\, dx} = -\frac{1}{4} Q^2(\psi_{t_1,t_1}).$$

This implies $I(\psi) \ge \inf\{-\frac{1}{4} Q^2(\phi) \mid \phi \in \mathcal{E}_{k,1}\}$. By (ii) we know that E_{GL} is bounded on $\mathcal{E}_{k,1}$ and Corollary 4.18 implies that Q is also bounded on $\mathcal{E}_{k,1}$. We conclude that $I_{min}(k) > -\infty$, hence $k < k_\infty$. Since this is true for any $k < T$, we infer that $k_\infty \ge T$. Thus $k_\infty = T$. ☐

Lemma 5.5 *Assume that $0 < k < k_\infty$ and $(\psi_n)_{n\ge1} \subset \mathcal{E}$ is a sequence such that $\int_{\mathbf{R}^N} |\nabla\psi_n|^2\, dx \le k$ for all n. Suppose that $(I(\psi_n))_{n\ge1}$ is bounded in the case $N \ge 3$, respectively that $I(\psi_n) < 0$ for all n in the case $N = 2$.*

Then $(Q(\psi_n))_{n\ge1}$, $\left(\int_{\mathbf{R}^N} V(|\psi_n|^2)\, dx\right)_{n\ge1}$ and $(E_{GL}(\psi_n))_{n\ge1}$ are bounded.

Proof. Consider first the case $N \ge 3$. Let us show that $\int_{\mathbf{R}^N} V(|\psi_n|^2)\, dx$ is bounded from above. We argue by contradiction and assume that this is false. Then there is a subsequence, still denoted $(\psi_n)_{n\ge1}$, such that $s_n := \int_{\mathbf{R}^N} V(|\psi_n|^2)\, dx \longrightarrow \infty$ as $n \longrightarrow \infty$.

Let $\sigma_n = s_n^{-\frac{1}{N}}$. Since $\int_{\mathbf{R}^N} |\nabla((\psi_n)_{\sigma_n,\sigma_n})|^2\, dx = \sigma_n^{N-2} \int_{\mathbf{R}^N} |\nabla\psi_n|^2\, dx \longrightarrow 0$ as $n \longrightarrow \infty$, Lemma 5.1 implies that $(\psi_n)_{\sigma_n,\sigma_n}$ satisfies (5.3) with $c = \frac{v_s}{2}$ for all sufficiently large n, that is

$$\int_{\mathbf{R}^N} |\nabla\psi_n|^2\, dx - \frac{v_s}{2} s_n^{-\frac{1}{N}} (s_n - I(\psi_n)) + s_n^{1-\frac{2}{N}} \ge 0.$$

Since $\int_{\mathbf{R}^N} |\nabla\psi_n|^2\, dx$ and $I(\psi_n)$ are bounded and $s_n \longrightarrow \infty$, the left-hand side of the above inequality tends to $-\infty$ as $n \longrightarrow \infty$, a contradiction. We conclude that there is $M > 0$ such that $\int_{\mathbf{R}^N} V(|\psi_n|^2)\, dx \le M$ for all M. Then (4.1) implies that $\int_{\mathbf{R}^N} \left(\varphi^2(|\psi_n|) - 1\right)^2\, dx$ is bounded. By (4.1), $\int_{\mathbf{R}^N} V(|\psi|^2)\, dx$ is bounded from below. Using Corollary 4.18 we infer that $(Q(\psi_n))_{n\ge1}$ is bounded.

Consider next the case $N = 2$. Since $\int_{\mathbf{R}^2} |\nabla\psi_n|^2\, dx \le k < k_\infty$, using Lemma 5.4 (i) and (iii) we see that either $\int_{\mathbf{R}^N} V(|\psi_n|^2)\, dx > 0$ or $\int_{\mathbf{R}^N} V(|\psi_n|^2)\, dx = 0$ and $|\psi_n| = 1$ a.e. on $\mathbf{R}^2$. In the latter case (2.7) implies $Q(\psi_n) = 0$, hence $I(\psi_n) = 0$, contrary to the assumption that $I(\psi_n) < 0$. Thus necessarily $0 < \int_{\mathbf{R}^N} V(|\psi_n|^2)\, dx < Q(\psi_n)$ for all n because $I(\psi_n) < 0$.

Since $\int_{\mathbf{R}^2} |\nabla(\psi_n)_{\sigma,\sigma}|^2\, dx = \int_{\mathbf{R}^2} |\nabla\psi_n|^2\, dx$ for any $\sigma > 0$, as in the proof of Lemma 5.4 (iii) we have

$$-\frac{Q^2(\psi_n)}{4\int_{\mathbf{R}^2} V(|\psi_n|^2)\, dx} = \inf_{\sigma>0} I((\psi_n)_{\sigma,\sigma}) \ge I_{min}\left(\int_{\mathbf{R}^2} |\nabla\psi_n|^2\, dx\right) \ge I_{min}(k)$$

and this implies

$$Q^2(\psi_n) \le -4I_{min}(k) \int_{\mathbf{R}^N} V(|\psi_n|^2)\, dx.$$

Combining this with the inequality $0 < \int_{\mathbf{R}^2} V(|\psi_n|^2)\, dx < Q(\psi_n)$, we get

$$0 < \int_{\mathbf{R}^2} V(|\psi_n|^2)\, dx < Q(\psi_n) \le -4I_{min}(k). \tag{5.12}$$

We have thus proved that $(Q(\psi_n))_{n\geq 1}$ and $\left(\int_{\mathbf{R}^2} V(|\psi_n|^2)\,dx\right)_{n\geq 1}$ are bounded. The boundedness of $\int_{\mathbf{R}^2} \left(\varphi^2(|\psi_n|) - 1\right)^2\,dx$ follows from Lemma 4.8 if $V \geq 0$ on $[0,\infty)$, respectively from Lemma 5.4 (ii) if V achieves negative values. $\square$

We now state the main result of this section, which shows precompactness of minimizing sequences for $I_{min}(k)$ as soon as $k_0 < k < k_\infty$.

Theorem 5.6 *Assume that $N \geq 2$ and (A1), (A2) hold. Let $k \in (k_0, k_\infty)$ and let $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ be a sequence such that*

$$\int_{\mathbf{R}^N} |\nabla \psi_n|^2\,dx \longrightarrow k \qquad \textit{and} \qquad I(\psi_n) \longrightarrow I_{min}(k).$$

There exist a subsequence $(\psi_{n_k})_{k\geq 1}$, a sequence of points $(x_k)_{k\geq 1} \subset \mathbf{R}^N$, and $\psi \in \mathcal{E}$ such that $\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx = k$, $I(\psi) = I_{min}(k)$, $\psi_{n_k}(x_k + \cdot) \longrightarrow \psi$ a.e. on $\mathbf{R}^N$ and

$$\|\nabla\psi_{n_k}(\cdot + x_k) - \nabla\psi\|_{L^2(\mathbf{R}^N)} \longrightarrow 0, \qquad \|\,|\psi_{n_k}|(\cdot + x_k) - |\psi|\,\|_{L^2(\mathbf{R}^N)} \longrightarrow 0 \qquad \textit{as } k \longrightarrow \infty.$$

Proof. Since $I_{min}(k) < 0$, we have $I(\psi_n) < 0$ for all sufficiently large n. By Lemma 5.5 the sequences $(Q(\psi_n))_{n\geq 1}$, $\left(\int_{\mathbf{R}^N} V(|\psi_n|^2)\,dx\right)_{n\geq 1}$ and $(E_{GL}(\psi_n))_{n\geq 1}$ are bounded. Passing to a subsequence if necessary, we may assume that $E_{GL}(\psi_n) \longrightarrow \alpha_0 \geq k > 0$ and $Q(\psi_n) \longrightarrow q$ as $n \longrightarrow \infty$.

We use the Concentration-Compactness Principle ([39]) and we argue as in the proof of Theorem 4.9. Let $\Lambda_n(t)$ be the concentration function associated to $E_{GL}(\psi_n)$, as in (4.15). It is standard to prove that there exist a subsequence of $((\psi_n, \Lambda_n))_{n\geq 1}$, still denoted $((\psi_n, \Lambda_n))_{n\geq 1}$, a nondecreasing function $\Lambda : [0,\infty) \longrightarrow \mathbf{R}$, $\alpha \in [0, \alpha_0]$, and a nondecreasing sequence $t_n \longrightarrow \infty$ such that (4.16) and (4.17) hold. The next result implies that $\alpha > 0$.

Lemma 5.7 *Let $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ be a sequence satisfying:*

(a) $E_{GL}(\psi_n) \leq M$ for some positive constant M.

(b) $\int_{\mathbf{R}^N} |\nabla\psi_n|^2\,dx \longrightarrow k$ and $Q(\psi_n) \longrightarrow q$ as $n \longrightarrow \infty$.

(c) $\limsup\limits_{n\to\infty} I(\psi_n) < -\frac{1}{v_s^2}k$.

Then there exists $\ell > 0$ such that $\sup\limits_{y\in\mathbf{R}^N} E_{GL}^{B(y,1)}(\psi_n) \geq \ell$ for all sufficiently large n.

Proof. It is obvious that the sequence $(\psi_n)_{n\geq 1}$ satisfies the conclusion of Lemma 5.7 if and only if $((\psi_n)_{v_s,v_s})_{n\geq 1}$ satisfies the same conclusion.

By (a) we have $E_{GL}((\psi_n)_{v_s,v_s}) \leq \max(v_s^{N-2}, v_s^N)M = 2^{\frac{N}{2}}M$. Assumption (b) implies

$$\int_{\mathbf{R}^N} |\nabla(\psi_n)_{v_s,v_s}|^2\,dx \longrightarrow v_s^{N-2}k \qquad \text{and} \qquad Q((\psi_n)_{v_s,v_s}) \longrightarrow v_s^{N-1}q = \tilde{q}.$$

Using (c) we find

$$\limsup_{n\to\infty} E((\psi_n)_{v_s,v_s}) - v_s\tilde{q} = \limsup_{n\to\infty}\left(v_s^{N-2}\int_{\mathbf{R}^N} |\nabla\psi_n|^2\,dx + v_s^N\int_{\mathbf{R}^N} V(|\psi_n|^2)\,dx - v_s^N Q(\psi_n)\right)$$
$$= v_s^N\left(\frac{1}{v_s^2}k + \limsup_{n\to\infty} I(\psi_n)\right) < 0.$$

Then the result follows directly from Lemma 4.10. $\square$

Next we prove that $\alpha \notin (0, \alpha_0)$. We argue again by contradiction and we assume that $0 < \alpha < \alpha_0$. Arguing as in the proof of Theorem 4.9 and using Lemma 3.3 for each n sufficiently large we construct two functions $\psi_{n,1}, \psi_{n,2} \in \mathcal{E}$ such that

$$E_{GL}(\psi_{n,1}) \longrightarrow \alpha \qquad \text{and} \qquad E_{GL}(\psi_{n,2}) \longrightarrow \alpha_0 - \alpha, \tag{5.13}$$

$$\int_{\mathbf{R}^N} \left| |\nabla \psi_n|^2 - |\nabla \psi_{n,1}|^2 - |\nabla \psi_{n,2}|^2 \right| dx \longrightarrow 0, \tag{5.14}$$

$$\int_{\mathbf{R}^N} \left| V(|\psi_n|^2) - V(|\psi_{n,1}|^2) - V(|\psi_{n,2}|^2) \right| dx \longrightarrow 0, \tag{5.15}$$

$$|Q(\psi_n) - Q(\psi_{n,1}) - Q(\psi_{n,2})| \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{5.16}$$

Passing to a subsequence if necessary, we may assume that $\int_{\mathbf{R}^N} |\nabla \psi_{n,i}|^2 \, dx \longrightarrow k_i \geq 0$ as $n \longrightarrow \infty$, $i = 1, 2$. By (5.14) we have $k_1 + k_2 = k$. We claim that $k_1 > 0$ and $k_2 > 0$.

To prove the claim assume, for instance, that $k_1 = 0$. From (5.13) it follows that $\frac{1}{2} \int_{\mathbf{R}^N} \left(\varphi^2(|\psi_{n,1}|) - 1 \right)^2 dx \longrightarrow \alpha$. Using Lemma 4.1 we find $\int_{\mathbf{R}^N} V(|\psi_{n,1}|^2) \, dx \longrightarrow \alpha$. From Lemma 4.4 (ii) we infer that there is $\kappa > 0$ such that $E(\psi) \geq \frac{v_s}{2} |Q(\psi)|$ for any $\psi \in \mathcal{E}$ satisfying $E_{GL}(\psi) \leq \kappa$. It is clear that there are $n_0 \in \mathbf{N}$ and $\sigma_0 > 0$ such that $E_{GL}((\psi_{n,1})_{\sigma,\sigma}) \leq \kappa$ for any $n \geq n_0$ and any $\sigma \in (0, \sigma_0]$. Then $E((\psi_{n,1})_{\sigma,\sigma}) \geq \frac{v_s}{2} |Q((\psi_{n,1})_{\sigma,\sigma})|$, that is

$$\frac{v_s}{2} |Q(\psi_{n,1})| \leq \frac{1}{\sigma} \int_{\mathbf{R}^N} |\nabla \psi_{n,1}|^2 \, dx + \sigma \int_{\mathbf{R}^N} V(|\psi_{n,1}|^2) \, dx$$

for any $n \geq n_0$ and $\sigma \in (0, \sigma_0]$. Passing to the limit as $n \longrightarrow \infty$ in the above inequality we discover $\frac{v_s}{2} \limsup\limits_{n \to \infty} |Q(\psi_{n,1})| \leq \sigma \alpha$ for any $\sigma \in (0, \sigma_0]$, that is $\lim\limits_{n \to \infty} |Q(\psi_{n,1})| = 0$. As a consequence we find $\lim\limits_{n \to \infty} I(\psi_{n,1}) = \alpha$. Since $|I(\psi_n) - I(\psi_{n,1}) - I(\psi_{n,2})| \longrightarrow 0$ by (5.15) and (5.16), we infer that $I(\psi_{n,2}) \longrightarrow I_{min}(k) - \alpha$ as $n \longrightarrow \infty$. Since $\int_{\mathbf{R}^N} |\nabla \psi_{n,2}|^2 \, dx \longrightarrow k_2 = k$, this contradicts the definition of I_{min} and the fact that I_{min} is continuous at k. Thus necessarily $k_1 > 0$. Similarly we have $k_2 > 0$, that is k_1, $k_2 \in (0, k)$.

We have $I(\psi_{n,i}) \geq I_{min}(\int_{\mathbf{R}^N} |\nabla \psi_{n,i}|^2 \, dx)$ and passing to the limit we get $\liminf\limits_{n \to \infty} I(\psi_{n,i}) \geq I_{min}(k_i)$, $i = 1, 2$. Using (5.15), (5.16) and the fact that $I(\psi_n) \longrightarrow I_{min}(k)$ we infer that $I_{min}(k) \geq I_{min}(k_1) + I_{min}(k_2)$. On the other hand, the concavity of I_{min} implies $I_{min}(k_i) \geq \frac{k_i}{k} I_{min}(k)$, hence $I_{min}(k_1) + I_{min}(k_2) \geq I_{min}(k)$ and equality may occur if and only if I_{min} is linear on $[0, k]$. Thus there is $A \in \mathbf{R}$ such that $I_{min}(s) = As$ for any $s \in [0, k]$. By Lemma 5.2 we have $A = -\frac{1}{v_s^2}$, hence $I_{min}(k) = -\frac{k}{v_s^2}$, contradicting the fact that $k > k_0$. Thus we cannot have $\alpha \in (0, \alpha_0)$, and then necessarily $\alpha = \alpha_0$.

As in the proof of Theorem 4.9, there is a sequence $(x_n)_{n \geq 1} \subset \mathbf{R}^N$ such that for any $\varepsilon > 0$ there is $R_\varepsilon > 0$ satisfying $E_{GL}^{\mathbf{R}^N \setminus B(x_n, R_\varepsilon)}(\psi_n) < \varepsilon$ for all n sufficiently large. Let $\tilde{\psi}_n = \psi_n(\cdot + x_n)$. Then for any $\varepsilon > 0$ there exist $R_\varepsilon > 0$ and $n_\varepsilon \in \mathbf{N}$ such that $(\tilde{\psi}_n)_{n \geq 1}$ satisfies (4.31). It is standard to prove that there is a function $\psi \in H^1_{loc}(\mathbf{R}^N)$ such that $\nabla \psi \in L^2(\mathbf{R}^N)$ and a subsequence $(\tilde{\psi}_{n_j})_{j \geq 1}$ satisfying (4.32)-(4.34) and (4.37).

Lemmas 4.11 and 4.12 imply that $\| \, |\tilde{\psi}_{n_j}| - |\psi| \, \|_{L^2(\mathbf{R}^N)} \longrightarrow 0$, $Q(\tilde{\psi}_{n_j}) \longrightarrow Q(\psi)$ and $\int_{\mathbf{R}^N} V(|\tilde{\psi}_{n_j}|^2) \, dx \longrightarrow \int_{\mathbf{R}^N} V(|\psi|^2) \, dx$ as $j \longrightarrow \infty$. Therefore $I(\tilde{\psi}_{n_j}) \longrightarrow I(\psi)$, and consequently $I(\psi) = I_{min}(k)$. On the other hand, by (4.33) we have $\int_{\mathbf{R}^N} |\nabla \psi|^2 \, dx \leq k$. Since I_{min} is strictly decreasing, we infer that necessarily $\int_{\mathbf{R}^N} |\nabla \psi|^2 \, dx = k = \lim\limits_{j \to \infty} \int_{\mathbf{R}^N} |\nabla \tilde{\psi}_{n_j}|^2 \, dx$. Combined with the weak convergence $\nabla \tilde{\psi}_{n_j} \rightharpoonup \nabla \psi$ in $L^2(\mathbf{R}^N)$, this gives the strong convergence $\nabla \tilde{\psi}_{n_j} \longrightarrow \nabla \psi$ in $L^2(\mathbf{R}^N)$ and the proof of Theorem 5.6 is complete. $\square$

Denote by $d^- I_{min}(k)$ and $d^+ I_{min}(k)$ the left and right derivatives of I_{min} at $k > 0$ (which exist and are finite for any $k > 0$ because I_{min} is concave). We have:

Proposition 5.8 *(i) Let $c > 0$. Then the function ψ is a minimizer of I in the set $\{\phi \in \mathcal{E} \mid \int_{\mathbf{R}^N} |\nabla\phi|^2\,dx = k\}$ if and only if $\psi_{c,c}$ minimizes the functional*

$$I_c(\phi) = -cQ(\phi) + \int_{\mathbf{R}^N} V(|\phi|^2)\,dx$$

in the set $\{\phi \in \mathcal{E} \mid \int_{\mathbf{R}^N} |\nabla\phi|^2\,dx = c^{N-2}k\}$.

(ii) If $\psi \in \mathcal{E}$ satisfies $\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx = k$ and $I(\psi) = I_{min}(k)$, there is $\vartheta \in [d^+ I_{min}(k), d^- I_{min}(k)]$ such that

$$i\psi_{x_1} - \vartheta\Delta\psi + F(|\psi|^2)\psi = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{5.17}$$

Then for $c = \frac{1}{\sqrt{-\vartheta}}$ the function $\psi_{c,c}$ satisfies (4.54) and minimizes $E_c = E - cQ$ in the set $\{\phi \in \mathcal{E} \mid \int_{\mathbf{R}^N} |\nabla\phi|^2\,dx = c^{N-2}k\}$. Moreover, $\psi \in W^{2,p}_{loc}(\mathbf{R}^N)$ and $\nabla\psi \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2,\infty)$.

(iii) After a translation, ψ is axially symmetric with respect to the x_1−axis if $N \geq 3$. The same conclusion is true if $N = 2$ and we assume in addition that F is C^1.

(iv) For any $k \in (k_0, k_\infty)$ there are $\psi^+, \psi^- \in \mathcal{E}$ such that $\int_{\mathbf{R}^N} |\nabla\psi^+|^2\,dx = \int_{\mathbf{R}^N} |\nabla\psi^-|^2\,dx = k$, $I(\psi^+) = I(\psi^-) = I_{min}(k)$ and ψ^+, ψ^- satisfy (5.17) with $\vartheta^+ = d^+ I_{min}(k)$ and $\vartheta^- = d^- I_{min}(k)$, respectively.

Proof. For any $\phi \in \mathcal{E}$ we have $I_c(\phi_{c,c}) = c^N I(\phi)$, $\int_{\mathbf{R}^N} |\nabla\phi_{c,c}|^2\,dx = c^{N-2}\int_{\mathbf{R}^N} |\nabla\phi|^2\,dx$ and (i) follows. The proofs of (ii), (iii) and (iv) are very similar to the proof of Proposition 4.14 and we omit them. □

We will establish later (see Proposition 8.4 below) a relationship between the traveling waves constructed in section 4 and those given by Theorem 5.6 and Proposition 5.8 above. The next remark shows that, in some sense, there is equivalence between the inequalities $E_{min}(q) < v_s q$ and $I_{min}(k) < -\frac{k}{v_s^2}$.

Remark 5.9 *(i)* Let $\psi \in \mathcal{E}$ be such that $E(\psi) < v_s Q(\psi)$ and let $k = \int_{\mathbf{R}^N} |\nabla\psi|^2\,dx$. Then $I_{min}(\frac{k}{v_s^{N-2}}) < -\frac{k}{v_s^N}$. Indeed, we have $\int_{\mathbf{R}^N} |\nabla\psi_{\frac{1}{v_s},\frac{1}{v_s}}|^2\,dx = \frac{1}{v_s^{N-2}}k$ and

$$I_{min}\left(\frac{k}{v_s^{N-2}}\right) + \frac{k}{v_s^N} \leq I\left(\psi_{\frac{1}{v_s},\frac{1}{v_s}}\right) + \frac{1}{v_s^2}\int_{\mathbf{R}^N} |\nabla\psi_{\frac{1}{v_s},\frac{1}{v_s}}|^2\,dx = \frac{1}{v_s^N}(E(\psi) - v_s Q(\psi)) < 0.$$

(ii) Conversely, let $\psi \in \mathcal{E}$ be such that $I(\psi) < -\frac{1}{v_s^2}\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx$ and denote $q = v_s^{N-1}Q(\psi)$. Then $E_{min}(p) < v_s q$. Indeed, we have $Q(\psi_{v_s,v_s}) = v_s^{N-1}Q(\psi) = q$ and

$$E_{min}(q) - v_s q \leq E(\psi_{v_s,v_s}) - Q(\psi_{v_s,v_s}) = v_s^N\left(\frac{1}{v_s^2}\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx + I(\psi)\right) < 0.$$

6 Local minimizers of the energy at fixed momentum ($N = 2$)

We will use the results in the previous section to find traveling waves to (1.1) in space dimension $N = 2$ which are *local* minimizers of the energy at fixed momentum even when V achieves negative values.

If $N = 2$ and $q \geq 0$, define

$$E^\sharp_{min}(q) = \inf\left\{E(\psi) \;\middle|\; \psi \in \mathcal{E},\ Q(\psi) = q \text{ and } \int_{\mathbf{R}^2} V(|\psi|^2)\,dx \geq 0\right\}. \tag{6.1}$$

This definition agrees with the one given in section 4 in the case $V \geq 0$.

Lemma 6.1 *Assume that $N = 2$ and (A1), (A2) are satisfied. The function $E^\sharp_{min}$ has the following properties:*

(i) $E^\sharp_{min}(q) \le v_s q$ for any $q \ge 0$.

(ii) For any $\varepsilon > 0$ there is $q_\varepsilon > 0$ such that $E^\sharp_{min}(q) > (v_s - \varepsilon)q$ for any $q \in (0, q_\varepsilon)$.

(iii) $E^\sharp_{min}$ is subadditive on $[0, \infty)$, nondecreasing, Lipschitz continuous and its best Lipschitz constant is v_s.

(iv) If $\inf V < 0$, then for any $q > 0$ we have $E^\sharp_{min}(q) \le k_\infty$, where k_∞ is as in (5.10) or in Lemma 5.4 (iii).

(v) $E^\sharp_{min}$ is concave on $[0, \infty)$.

Proof. If $V \ge 0$ on $[0, \infty)$, the statements of Lemma 6.1 have already been proven in section 4. We only consider here the case when V achieves negative values. The estimate (i) follows from Lemma 4.5. For (ii) proceed as in the proof of Lemma 4.6 and use Lemma 5.1 instead of Lemma 4.4. The proof of (iii) is the same as that of Lemma 4.7 (i).

(iv) Let $q > 0$. Fix $\varepsilon > 0$, ε small. By (ii) there is $\psi \in \mathcal{E}$ such that $\int_{\mathbf{R}^2} |\nabla\psi|^2\,dx \le \frac{\varepsilon}{4}$ and $Q(\psi) \ge \frac{\varepsilon}{8v_s}$. It is obvious that $\int_{\mathbf{R}^2} |\nabla(\psi_{\sigma,\sigma})|^2\,dx = \int_{\mathbf{R}^2} |\nabla\psi|^2\,dx \le \frac{\varepsilon}{4}$ for any $\sigma > 0$ and there is $\sigma_0 > 0$ such that $Q(\psi_{\sigma_0,\sigma_0}) > q$. Using Corollary 3.4 and (2.12), we see that there is $\psi_1 \in \mathcal{E}$ such that $Q(\psi_1) = q$, $\int_{\mathbf{R}^2} |\nabla\psi_1|^2\,dx \le \frac{\varepsilon}{2}$ and $\psi_1 = 1$ outside a large ball $B(0, R_1)$. Let $M_1 = \int_{\mathbf{R}^2} V(|\psi_1|^2)\,dx$.

Let ψ_0 be as in Lemma 5.4 (i). Proceeding as in the proof of Lemma 5.4 (iii) we see that there exists a radial function $\phi \in C^\infty_c(\mathbf{R}^2)$ and there is $\varepsilon_1 > 0$ such that $\int_{\mathbf{R}^2} V(|\psi_0 + t\phi|^2)\,dx < 0$ for any $t \in (0, \varepsilon_1)$. Taking $t \in (0, \varepsilon_1)$ sufficiently small and using a radial cut-off and scaling it is not hard to construct a radial function $\psi_2 \in \mathcal{E}$ such that $\int_{\mathbf{R}^2} |\nabla\psi_2|^2\,dx \le k_\infty + \frac{\varepsilon}{4}$, $\int_{\mathbf{R}^2} V(|\psi_2|^2)\,dx = -M_2 < 0$ and $\psi_2 = 1$ outside a large ball $B(0, R_2)$. Since ψ_2 is radial, we have $Q(\psi_2) = 0$.

Let $t = \left(\frac{M_1 - \frac{\varepsilon}{4}}{M_2}\right)^{\frac{1}{2}}$. Choose $x_0 \in \mathbf{R}^2$ such that $|x_0| > 2(R_1 + tR_2)$ and define

$$\psi_*(x) = \begin{cases} \psi_1(x) & \text{if } |x| \le R_1, \\ \psi_2\left(\frac{x - x_0}{t}\right) & \text{if } |x| > R_1. \end{cases}$$

Then $\psi_* \in \mathcal{E}$, $Q(\psi_*) = Q(\psi_1) + tQ(\psi_2) = q$, $\int_{\mathbf{R}^2} |\nabla\psi_*|^2\,dx = \int_{\mathbf{R}^2} |\nabla\psi_1|^2\,dx + \int_{\mathbf{R}^2} |\nabla\psi_2|^2\,dx \le k_\infty + \frac{3\varepsilon}{4}$, and $\int_{\mathbf{R}^2} V(|\psi_*|^2)\,dx = \int_{\mathbf{R}^2} V(|\psi_1|^2)\,dx + t^2 \int_{\mathbf{R}^2} V(|\psi_2|^2)\,dx = M_1 - t^2 M_2 = \frac{\varepsilon}{4} > 0$. Thus $E^\sharp_{min}(q) \le E(\psi_*) \le k_\infty + \varepsilon$. Since ε is arbitrary, the conclusion follows.

(v) The idea is basically the same as in the proof of Lemma 4.7 (ii) but we have to be more careful because the functions $\psi \in \mathcal{E}$ that satisfy $\int_{\mathbf{R}^2} V(|\psi|^2)\,dx \ge 0$ do not necessarily satisfy $\int_{\mathbf{R}^2} V(|S^\pm_t\psi|^2)\,dx \ge 0$ for all t, where $S^\pm_t$ are as in (4.10) - (4.11).

Let $E^\sharp = \sup\limits_{q \ge 0} E^\sharp_{min}(q)$. By (iv) we have $E^\sharp \le k_\infty$. Denote

$$q^\sharp = \sup\{q > 0 \mid E^\sharp_{min}(q) < E^\sharp\}. \tag{6.2}$$

Define $E^{\sharp,-1}_{min}(k) = \sup\{q \ge 0 \mid E^\sharp_{min}(q) \le k\}$. Then $E^{\sharp,-1}_{min}$ is finite, increasing, right continuous on $[0, E^\sharp)$ and $E^\sharp_{min}(E^{\sharp,-1}_{min}(k)) = k$ for all $k \in [0, E^\sharp)$. By convention, put $E^{\sharp,-1}_{min}(k) = 0$ if $k < 0$. For any $\phi \in \mathcal{E}$ with $\int_{\mathbf{R}^2} V(|\phi|^2)\,dx \ge 0$ we have $E^\sharp_{min}(Q(\phi)) \le E(\phi)$, thus

$$Q(\phi) \le E^{\sharp,-1}_{min}(E(\phi)). \tag{6.3}$$

We will prove that for any fixed $q \in (0, q^\sharp)$ there are $q_1 < q$ and $q_2 > q$ such that $E^\sharp_{min}$ is concave on $[q_1, q_2]$.

Let $q \in (0, q^\sharp)$. Fix an arbitrary $\varepsilon > 0$ such that $E^\sharp_{min}(q) + 4\varepsilon < E^\sharp$. Choose $\psi \in \mathcal{E}$ such that $Q(\psi) = q$, $\int_{\mathbf{R}^2} V(|\psi|^2)\, dx > 0$ and $E(\psi) < E^\sharp_{min}(q) + \varepsilon$.

We may assume that ψ is symmetric with respect to x_2. Indeed, let S^+_t and S^-_t be as in (4.10)-(4.11). Arguing as in the proof of Lemma 4.7 (ii), there is $t_0 \in \mathbf{R}$ such that $\int_{\mathbf{R}^2} |\nabla(S^+_{t_0}(\psi))|^2\, dx = \int_{\mathbf{R}^2} |\nabla(S^-_{t_0}(\psi))|^2\, dx = \int_{\mathbf{R}^2} |\nabla\psi|^2\, dx < k_\infty$. After a translation, we may assume that $t_0 = 0$. Let $\psi_1 = S^-_0(\psi)$, $\psi_2 = S^+_0(\psi)$, denote $q_i = Q(\psi_i)$ and $v_i = \int_{\mathbf{R}^2} V(|\psi_i|^2)\, dx$, $i = 1, 2$ and $v = \int_{\mathbf{R}^2} V(|\psi|^2)$, so that $q_1 + q_2 = 2Q(\psi) = 2q$ and $v_1 + v_2 = 2v$. Since $\int_{\mathbf{R}^2} |\nabla\psi_i)|^2\, dx < k_\infty = T$, by Lemma 5.4 we have $v_1 \geq 0$ and $v_2 \geq 0$ and consequently $v_1, v_2 \in [0, 2v]$. If $q_1 \leq 0$ we have $q_2 \geq 2q$ and then for $\sigma_2 = \frac{q}{q_2} \leq \frac{1}{2}$, we get $Q((\psi_2)_{\sigma_2,\sigma_2}) = q$ and $E((\psi_2)_{\sigma_2,\sigma_2}) \leq E(\psi) < E^\sharp_{min}(q) + \varepsilon$, hence we may choose $(\psi_2)_{\sigma_2,\sigma_2}$ instead of ψ, and $(\psi_2)_{\sigma_2,\sigma_2}$ is symmetric with respect to x_2. A similar argument works if $q_2 \leq 0$. If $q_1 > 0$ and $q_2 > 0$, let $\sigma_1 = \frac{q}{q_1}$ and $\sigma_2 = \frac{q}{q_2}$, so that $\frac{1}{\sigma_1} + \frac{1}{\sigma_2} = 2$. We claim that there is $i \in \{1, 2\}$ such that $\sigma_i^2 v_i \leq v$, and then we may choose $(\psi_i)_{\sigma_i,\sigma_i}$, which is symmetric with respect to x_2, instead of ψ . Indeed, if the claim is false we have $v_i > \frac{1}{\sigma_i^2} v$ and taking the sum we get $2 > \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$, which is impossible because $\frac{1}{\sigma_1} + \frac{1}{\sigma_2} = 2$.

Since ψ is symmetric with respect to x_2, we have $Q(S^\pm_0\psi) = q$ and $E(S^\pm_0\psi) = E(\psi) < k_\infty - 3\varepsilon$. As in Lemma 4.7 (ii), the mapping $t \longmapsto E(S^-_t\psi)$ is continuous and tends to $2E(\psi)$ as $t \longrightarrow \infty$. Let

$$t_\infty = \inf\{t \geq 0 \mid E(S^-_t\psi) \geq k_\infty\} \qquad \text{(with possibly } t_\infty = \infty \text{ if } E(\psi) \leq \frac{1}{2}k_\infty).$$

For any $t \in [0, t_\infty)$ we have $E(S^-_t\psi) < k_\infty$. If there is $t \in [0, t_\infty)$ such that $\int_{\mathbf{R}^2} V(|S^-_t\psi|^2)\, dx = 0$, we have necessarily $\int_{\mathbf{R}^2} |\nabla(S^-_t\psi)|^2\, dx \geq k_\infty$, thus $E(S^-_t\psi) \geq k_\infty$, a contradiction. We infer that the function $t \longmapsto \int_{\mathbf{R}^2} V(|S^-_t\psi|^2)\, dx$ is continuous, positive at $t = 0$ and cannot vanish on $[0, t_\infty)$, hence $\int_{\mathbf{R}^2} V(|S^-_t\psi|^2)\, dx > 0$ for all $t \in [0, t_\infty)$. Consequently we have

$$E(S^-_t\psi) \geq E^\sharp_{min}(Q(S^-_t\psi)) \qquad \text{for any } t \in [0, t_\infty). \tag{6.4}$$

For any $t \geq 0$ we have $\int_{\mathbf{R}^2} |\nabla(S^+_t\psi)|^2\, dx = 2\int_{\{x_2 \geq t\}} |\nabla\psi|^2\, dx \leq 2\int_{\{x_2 \geq 0\}} |\nabla\psi|^2\, dx \leq E(\psi) < k_\infty$, hence $\int_{\mathbf{R}^2} V(|S^+_t\psi|^2)\, dx \geq 0$ (by Lemma 5.4) and therefore

$$E(S^+_t\psi) \geq E^\sharp_{min}(Q(S^+_t\psi)) \qquad \text{for any } t \geq 0. \tag{6.5}$$

The mapping $t \longmapsto Q(S^+_t\psi)$ is continuous, tends to 0 as $t \longrightarrow \infty$ and $Q(S^+_0\psi) = q$. If $t_\infty = \infty$, for any $q_1 \in (0, q)$ there is $t_{q_1} > 0$ such that $Q(S^+_{t_{q_1}}\psi) = q_1$. Then $Q(S^-_{t_{q_1}}\psi) = 2q - q_1$ and using (6.1), (6.4) we get

$$E^\sharp_{min}(q) + \varepsilon > E(\psi) = \frac{1}{2}\left(E(S^+_{t_{q_1}}\psi) + E(S^-_{t_{q_1}}\psi)\right) \geq \frac{1}{2}\left(E^\sharp_{min}(q_1) + E^\sharp_{min}(2q - q_1)\right).$$

In the case $t_\infty < \infty$ we have $E(S^-_{t_\infty}\psi) = k_\infty$, hence $E(S^+_{t_\infty}\psi) = 2E(\psi) - E(S^-_{t_\infty}\psi) < 2E^\sharp_{min}(q) + 2\varepsilon - k_\infty < E^\sharp_{min}(q)$ and by (6.3) it follows that

$$Q(S^+_{t_\infty}\psi) \leq E^{\sharp,-1}_{min}(2E^\sharp_{min}(q) + 2\varepsilon - k_\infty) < q.$$

For any $q_1 \in [Q(S^+_{t_\infty}\psi), q]$ there is $t_{q_1} \in [0, t_\infty]$ such that $Q(S^+_{t_{q_1}}\psi) = q_1$. As above, we obtain

$$E^\sharp_{min}(q) + \varepsilon > \frac{1}{2}\left(E^\sharp_{min}(q_1) + E^\sharp_{min}(2q - q_1)\right) \qquad \text{for any } q_1 \in [E^{\sharp,-1}_{min}(2E^\sharp_{min}(q) + 2\varepsilon - k_\infty),\ q].$$

Since $\varepsilon \in (0, \frac{1}{4}(E^\sharp - E^\sharp_{min}(q)))$ is arbitrary and $E^{\sharp,-1}_{min}$ is right continuous we infer that for any $q \in (0, q^\sharp)$ there holds

$$(6.6) \qquad E^\sharp_{min}(q) \geq \frac{1}{2}\Big(E^\sharp_{min}(q_1) + E^\sharp_{min}(2q - q_1)\Big) \qquad \text{for all } q_1 \in (E^{\sharp,-1}_{min}(2E^\sharp_{min}(q) - k_\infty),\ q].$$

The function $q \longmapsto E^{\sharp,-1}_{min}(2E^\sharp_{min}(q) - k_\infty)$ is nondecreasing and right continuous on $(0, q^\sharp)$. Fix $q_* \in (0, q^\sharp)$. We have

$$\lim_{q \downarrow q_*} E^{\sharp,-1}_{min}(2E^\sharp_{min}(q) - k_\infty) = E^{\sharp,-1}_{min}(2E^\sharp_{min}(q_*) - k_\infty) < q_*$$

because $2E^\sharp_{min}(q_*) - k_\infty < E^\sharp_{min}(q_*)$. It is then easy to see that there are $q_*' < q_*$ and $q_*'' \in (q_*, q^\sharp)$ such that for any $q \in [q_*', q_*'']$,

$$(6.7) \qquad E^{\sharp,-1}_{min}(2E^\sharp_{min}(q) - k_\infty) < q_*'.$$

Using (6.6) we see that for any $q_1, q_2 \in [q_*', q_*'']$ we have

$$E^\sharp_{min}\left(\frac{q_1 + q_2}{2}\right) \geq \frac{1}{2}(E^\sharp_{min}(q_1) + E^\sharp_{min}(q_2)).$$

Since $E^\sharp_{min}$ is continuous, we infer that $E^\sharp_{min}$ is concave on $[q_*', q_*'']$. Thus any point $q_* \in (0, q^\sharp)$ has a neighborhood where $E^\sharp_{min}$ is concave and then it is not hard to see that $E^\sharp_{min}$ is concave on $[0, q^\sharp)$. If $q^\sharp < \infty$ we have $E^\sharp_{min} = E^\sharp$ on $[q^\sharp, \infty)$, hence $E^\sharp_{min}$ is concave on $[0, \infty)$. $\square$

Let

$$(6.8) \qquad q^\sharp_0 = \inf\{q > 0 \mid E^\sharp_{min}(q) < v_s q\} \qquad \text{and} \qquad q^\sharp_\infty = \sup\{q > 0 \mid E^\sharp_{min}(q) < k_\infty\}.$$

It is obvious that $q^\sharp_0 \leq q^\sharp_\infty$ and $q_\infty > 0$ because $E^\sharp_{min}(q) \to 0 < k_\infty$ as $q \longrightarrow 0$. If F satisfies assumption (A4) and $F''(1) \neq 3$, it follows from Theorem 4.15 that $q^\sharp_0 = 0$ (notice that the test functions U_ε constructed in the proof of Theorem 4.15 satisfy $V(|U_\varepsilon|^2) \geq 0$ in $\mathbf{R}^2$).

Our next result shows the precompactness of minimizing sequences for $E^\sharp_{min}(q)$.

Theorem 6.2 *Assume that $N = 2$, (A1), (A2) are satisfied, and $\inf V < 0$. Let $q \in (q^\sharp_0, q^\sharp_\infty)$ and assume that $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ is a sequence satisfying*

$$\int_{\mathbf{R}^2} V(|\psi_n|^2)\, dx \geq 0, \quad Q(\psi_n) \longrightarrow q \quad \textit{and} \quad E(\psi_n) \longrightarrow E^\sharp_{min}(q).$$

There exist a subsequence $(\psi_{n_k})_{k\geq 1}$, a sequence of points $(x_k)_{k\geq 1} \subset \mathbf{R}^N$, and $\psi \in \mathcal{E}$ such that $Q(\psi) = q$, $E(\psi) = E^\sharp_{min}(q)$, $\psi_{n_k}(x_k + \cdot) \longrightarrow \psi$ a.e. on $\mathbf{R}^2$ and $\lim_{k\to\infty} d_0(\psi_{n_k}(x_k + \cdot),\ \psi) = 0$. Furthermore, $\int_{\mathbf{R}^2} V(|\psi|^2)\, dx > 0$, hence $\psi \in \mathcal{E}$ is a local minimizer in the sense that

$$E(\psi) = E^\sharp_{min}(q) = \inf\Big\{E(w) \;\Big|\; w \in \mathcal{E}, \qquad Q(w) = q, \qquad \int_{\mathbf{R}^2} V(|w|^2)\, dx > 0\Big\}.$$

Moreover, the conclusions of Proposition 4.14 hold true with E_{min} replaced by $E^\sharp_{min}$.

Proof. Fix k_1, k_2 such that $0 < k_1 < E^\sharp_{min}(q) < k_2 < k_\infty$. We may assume that $k_1 < E(\psi_n) < k_2$ for all n. By Lemma 4.1 there is $C_1(k_1) > 0$ such that $E_{GL}(\psi_n) \geq C_1(k_1)$. Since $\int_{\mathbf{R}^2} V(|\psi_n|^2)\, dx \geq 0$, we have $\psi_n \in \mathcal{E}_{k_2,k_2}$ and using Lemma 5.4 we infer that $E_{GL}(\psi_n)$

is bounded. Passing to a subsequence if necessary, we may assume that $E_{GL}(\psi_n) \longrightarrow \alpha_0 > 0$. Then we proceed as in the proof of Theorem 4.9 and we use the Concentration-Compactness Principle for the sequence of functions $f_n = |\nabla \psi_n|^2 + \frac{1}{2}\left(\varphi^2(|\psi_n|) - 1\right)^2$.

We rule out vanishing thanks to Lemma 4.10.

If dichotomy occurs for a subsequence (still denoted $(\psi_n)_{n\geq 1}$), using Lemma 3.3 for all n sufficiently large we construct two functions $\psi_{n,1},\ \psi_{n,2} \in \mathcal{E}$ such that $\left| \int_{\mathbf{R}^2} |\nabla \psi_n|^2\, dx - \int_{\mathbf{R}^2} |\nabla \psi_{n,1}|^2\, dx - \int_{\mathbf{R}^2} |\nabla \psi_{n,2}|^2\, dx \right| \longrightarrow 0$, and (4.28), (4.29), (4.30) hold for some $\alpha \in (0, \alpha_0)$. In particular, we have $\int_{\mathbf{R}^2} |\nabla \psi_{n,i}|^2\, dx < k_2 < k_\infty$, $i = 1, 2$ for all n sufficiently large and this implies $\int_{\mathbf{R}^2} V(|\psi_{n,i}|^2)\, dx \geq 0$, so that $E(\psi_{n,i}) \geq E^\sharp_{min}(Q(\psi_{n,i}))$. Since $q \in (q_0^\sharp, q_\infty^\sharp)$, using the concavity of $E^\sharp_{min}$ and Lemma 6.1 (i) and (ii) we infer that $E^\sharp_{min}(q) < E^\sharp_{min}(q') + E^\sharp_{min}(q - q')$ for any $q' \in (0, q)$. Then arguing as in the proof of Theorem 4.9 we rule out dichotomy and we conclude that concentration occurs.

Hence there is a sequence $(x_n)_{n\geq 1} \subset \mathbf{R}^N$ such that, denoting $\tilde{\psi}_n = \psi_n(x_n + \cdot)$, (4.31) holds. Consequently there are a subsequence $(\tilde{\psi}_{n_k})_{k\geq 1}$ and $\psi \in \mathcal{E}$ that satisfy (4.32) and (4.33). Using Lemmas 4.11 and 4.12 we get $\lim\limits_{k\to\infty} \| \, |\tilde{\psi}_{n_k}| - |\psi| \, \|_{L^2(\mathbf{R}^N)} = 0$,

$$\text{(6.9)} \qquad \lim_{k\to\infty} \int_{\mathbf{R}^2} V(|\tilde{\psi}_{n_k}|^2)\, dx = \int_{\mathbf{R}^2} V(|\psi|^2)\, dx \quad \text{and} \quad \lim_{k\to\infty} Q(\tilde{\psi}_{n_k}) = Q(\psi).$$

In particular, we have $\int_{\mathbf{R}^2} V(|\psi|^2)\, dx \geq 0$, $Q(\psi) = q$ and this implies $E(\psi) \geq E^\sharp_{min}(q)$. Combining this information with (4.33) and (6.9) we see that necessarily $\int_{\mathbf{R}^2} |\nabla \tilde{\psi}_{n_k}|^2\, dx \longrightarrow \int_{\mathbf{R}^2} |\nabla \psi|^2\, dx$. Together with the weak convergence $\nabla \tilde{\psi}_{n_k} \rightharpoonup \nabla \psi$ in $L^2(\mathbf{R}^2)$, this implies the strong convergence $\|\nabla \tilde{\psi}_{n_k} - \nabla \psi\|_{L^2(\mathbf{R}^2)} \longrightarrow 0$. Hence $d_0(\tilde{\psi}_{n_k}, \psi) \longrightarrow 0$ as $k \longrightarrow \infty$. The fact that $\int_{\mathbf{R}^2} V(|\psi|^2)\, dx > 0$ comes from the fact that $|\psi|$ is not constant (because $Q(\psi) = q > 0$) and $\int_{\mathbf{R}^2} |\nabla \psi|^2\, dx < k_\infty$.

The last part is proved in the same way as Proposition 4.14. $\square$

If $q_\infty^\sharp < \infty$ we have $E^\sharp_{min}(q) = k_\infty$ for all $q \geq q_\infty^\sharp$. The conclusion of Theorem 6.2 is not valid for $q \geq q_\infty^\sharp$. Indeed, for such q the argument used in the proof of Lemma 6.1 (iv) leads to the construction of a minimizing sequence $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ satisfying the assumptions of Theorem 6.2, but $E_{GL}(\psi_n) \longrightarrow \infty$. Furthermore, if $\int_{\mathbf{R}^2} |\nabla \psi|^2\, dx \geq k_\infty$, Lemma 5.4 does not guarantee that the potential energy $\int_{\mathbf{R}^2} V(|\psi|^2)\, dx$ is positive.

7 Orbital stability

It is beyond the scope of the present paper to study the Cauchy problem associated to (1.1). Instead, we will content ourselves to assume in the sequel that the nonlinearity F satisfies (A1), (A2) and is such that the following holds:

(P1) (local well-posedness) For any $M > 0$ there is $T(M) > 0$ such that for any $\psi_0 \in \mathcal{E}$ with $E_{GL}(\psi_0) \leq M$ there exist $T_{\psi_0} \geq T(M)$ and a unique solution $t \longmapsto \psi(t) \in C([0, T_{\psi_0}), (\mathcal{E}, d))$ such that $\psi(0) = \psi_0$. Moreover, $\psi(\cdot)$ depends continuously on the initial data in the following sense: if $d(\psi_0^n, \psi_0) \longrightarrow 0$ and $t \longmapsto \psi_n(t)$ is the solution of (1.1) with initial data ψ_0^n, then for any $T < T_{\psi_0}$ we have $T < T_{\psi_0^n}$ for all sufficiently large n and $d(\psi_n(t), \psi(t)) \longrightarrow 0$ uniformly on $[0, T]$ as $n \longrightarrow \infty$.

(P2) (conservation of phase at infinity) We have $\psi(\cdot) - \psi_0 \in C([0, T_{\psi_0}), H^1(\mathbf{R}^N))$.

(P3) (conservation of energy) We have $E(\psi(t)) = E(\psi_0)$ for any $t \in [0, T_{\psi_0})$.

(P4) (regularity) If $\Delta\psi_0 \in L^2(\mathbf{R}^N)$, then $\Delta\psi(\cdot) \in C([0, T_{\psi_0}), L^2(\mathbf{R}^N))$.

In space dimension $N = 2, 3, 4$, the Cauchy problem for the Gross-Pitaevskii equation (that is (1.1) with $F(s) = 1 - s$) has been studied in [24, 25] and it was proved that the flow has the properties (P1)-(P4) above. Moreover, the solutions found in [24, 25] are global in time if $N = 2, 3$ or if $N = 4$ and the initial data has sufficiently small energy. This comes from the conservation of energy and from the fact that the Gross-Pitaevskii equation is subcritical if $N = 2$, 3 and it is critical if $N = 4$. It seems that the proofs in [24, 25] can be easily adapted to more general subcritical nonlinearities provided that the associated nonlinear potential V is nonnegative on $[0, \infty)$. Notice that any nonlinearity satisfying (A2) is subcritical.

Recently it has been proved in [34] that the Gross-Pitaevskii equation is globally well-posed on the whole energy space $\mathcal{E}$ in space dimension $N = 4$ and that the cubic-quintic NLS is globally well-posed on $\mathcal{E}$ if $N = 3$, despite the fact that both problems are critical.

Assume that (P1) and (P3) hold. If $V \geq 0$, using the conservation of energy and Lemma 4.8 it is easy to prove that all solutions are global.

If $N = 2$ and $\inf V < 0$, any solution $t \longmapsto \psi(t)$ with initial data ψ_0 satisfying $\int_{\mathbf{R}^2} |\nabla\psi_0|^2 \, dx < k_\infty$ and $E(\psi_0) < k_\infty$ is global. Indeed, the mapping $t \longmapsto \int_{\mathbf{R}^2} V(|\psi(t)|^2)dx$ is continuous; if it changes sign at some $t_0 \in (0, T_{\psi_0})$, there are two possibilities: either $\psi(t_0)$ is constant (and then $E(\psi(t_0)) = 0$, hence $E(\psi(t)) = 0$ for all t and $\psi(t)$ is constant) or Lemma 5.4 (i) implies that $\int_{\mathbf{R}^2} V(|\psi(t_0)|^2)dx = 0$ and $\int_{\mathbf{R}^2} |\nabla\psi_0|^2 \, dx \geq k_\infty$, thus $E(\psi(t_0)) \geq k_\infty$, contradicting the fact that, by conservation of the energy, $E(\psi(t_0)) = E(\psi_0) < k_\infty$. Consequently $0 \leq \int_{\mathbf{R}^2} V(|\psi(t)|^2)dx \leq E(\psi_0)$ and $0 \leq \int_{\mathbf{R}^2} |\nabla\psi(t)|^2 dx \leq E(\psi_0)$ as long as the solution exists. Then Lemma 5.4 (ii) implies that $E_{GL}(\psi(t))$ remains bounded and using (P1) we see that the solution is global.

In the case of more general nonlinearities, the Cauchy problem for (1.1) has been considered by C. Gallo in [23]. In space dimension $N = 1, 2, 3, 4$ and under suitable assumptions on F, he proved the following (see Theorems 1.1 and 1.2 pp. 731-732 in [23]):

(P1') For any $\psi_0 \in \mathcal{E}$ and any $u_0 \in H^1(\mathbf{R}^N)$, there exists a unique global solution $\psi_0 + u(t)$, where $u(\cdot) \in C([0, \infty), H^1(\mathbf{R}^N))$ and $u(0) = u_0$. The solution depends continuously on the initial data $u_0 \in H^1(\mathbf{R}^N)$.

Notice that the solutions in [23] satisfy (P2) by construction and they also satisfy (P3) and (P4). Moreover, it is proved (see Theorem 1.5 p. 733 in [23]) that any solution $\psi \in C([0, T], \mathcal{E})$ automatically satisfies (P2).

Lemma 7.1 *(conservation of the momentum) Assume that F is such that (A1), (A2), ((P1) or (P1')) and (P2)–(P4) hold. Let $\psi_0 \in \mathcal{E}$ and let ψ be the solution of (1.1) with initial data ψ_0, as given by (P1) or (P1')). Then*

$$Q(\psi(t)) = Q(\psi_0) \qquad \textit{for any } t \in [0, T_{\psi_0}).$$

Proof. Assume that $\psi_0 \in \mathcal{E}$ is such that $\Delta\psi_0 \in L^2(\mathbf{R}^N)$. Let $\psi(\cdot)$ be the solution of (1.1) with initial data ψ_0. By (P1) and (P4) we have $\psi_{x_j}(\cdot) \in C([0, T_{\psi_0}), H^1(\mathbf{R}^N))$, $j = 1, \dots, N$. Let $t, t+s \in [0, T_{\psi_0})$. Since $\psi(t+s) - \psi(t) \in H^1(\mathbf{R}^N)$ by (P2), the Cauchy-Schwarz inequality implies $\langle i\psi_{x_1}(t+s) + i\psi_{x_1}(t), \psi(t+s) - \psi(t)\rangle \in L^1(\mathbf{R}^N)$. Using the definition of the momentum and Lemma 2.3 we get

$$\tfrac{1}{s}\left(Q(\psi(t+s)) - Q(\psi(t))\right) = \tfrac{1}{s}L(\langle i\psi_{x_1}(t+s) + i\psi_{x_1}(t),\ \psi(t+s) - \psi(t)\rangle)$$

$$= \int_{\mathbf{R}^N} \langle i\psi_{x_1}(t+s) + i\psi_{x_1}(t),\ \tfrac{1}{s}(\psi(t+s) - \psi(t))\rangle \, dx.$$

Letting $s \longrightarrow 0$ in the above equality and using (1.1) we get

$$\frac{d}{dt}(Q(\psi(t))) = 2\int_{\mathbf{R}^N} \langle \frac{\partial \psi(t)}{\partial x_1},\ \Delta\psi(t) + F(|\psi|^2)\psi(t)\rangle\, dx. \tag{7.1}$$

Since $\frac{\partial \psi(t)}{\partial x_j} \in H^1(\mathbf{R}^N)$, using the integration by parts formula for H^1 functions (see, e.g., [10] p. 197) we have

$$\int_{\mathbf{R}^N} \langle \frac{\partial \psi(t)}{\partial x_1},\ \Delta\psi(t)\rangle\, dx = -\int_{\mathbf{R}^N} \sum_{j=1}^{N} \langle \frac{\partial^2 \psi(t)}{\partial x_1 \partial x_j},\ \frac{\partial \psi(t)}{\partial x_j}\rangle\, dx = -\frac{1}{2}\int_{\mathbf{R}^N} \frac{\partial}{\partial x_1}\left(|\nabla\psi(t)|^2\right)\, dx. \tag{7.2}$$

We have $|\nabla\psi(t)|^2 \in L^1(\mathbf{R}^N)$ and $\frac{\partial}{\partial x_k}\left(|\nabla\psi(t)|^2\right) = 2\sum_{j=1}^{N}\langle \frac{\partial^2\psi(t)}{\partial x_k\partial x_j},\ \frac{\partial\psi(t)}{\partial x_j}\rangle \in L^1(\mathbf{R}^N)$, hence $|\nabla\psi(t)|^2 \in W^{1,1}(\mathbf{R}^N)$. It is well-known that for any $f \in W^{1,1}(\mathbf{R}^N)$ we have $\int_{\mathbf{R}^N}\frac{\partial f}{\partial x_j}(x)\, dx = 0$ and using (7.2) we get $\int_{\mathbf{R}^N}\langle \frac{\partial\psi(t)}{\partial x_1},\ \Delta\psi(t)\rangle\, dx = 0$.

On the other hand, $2\langle \psi_{x_1}(t),\ F(|\psi|^2)\psi(t)\rangle = -\frac{\partial}{\partial x_1}\left(V(|\psi(t)|^2)\right)$. We have $V(|\psi(t)|^2) \in L^1(\mathbf{R}^N)$ by Lemma 4.1. Using the fact that $\psi_{x_j}(t) \in H^1(\mathbf{R}^N)$, (A1), (A2) and the Sobolev embedding it is easy to see that $\frac{\partial}{\partial x_j}\left(V(|\psi(t)|^2)\right) = -2\langle\psi_{x_j}(t),\ F(|\psi|^2)\psi(t)\rangle \in L^1(\mathbf{R}^N)$ for all j, hence $V(|\psi(t)|^2) \in W^{1,1}(\mathbf{R}^N)$ and therefore $\int_{\mathbf{R}^N}\frac{\partial}{\partial x_1}\left(V(|\psi(t)|^2)\right)\, dx = 0$. Then using (7.1) we obtain $\frac{d}{dt}(Q(\psi(t))) = 0$ for any t, consequently $Q(\psi(\cdot))$ is constant on $[0, T_{\psi_0})$.

Let $\psi_0 \in \mathcal{E}$ be arbitrary. By Lemma 3.5, there is a sequence $(\psi_0^n)_{n\geq 1} \subset \mathcal{E}$ such that $\nabla\psi_0^n \in H^2(\mathbf{R}^N)$ and $\|\psi_0^n - \psi_0\|_{H^1(\mathbf{R}^N)} \longrightarrow 0$ as $n \longrightarrow \infty$ (thus, in particular, $d(\psi_0^n, \psi_0) \longrightarrow 0$). Fix $T \in (0, T_{\psi_0})$. It follows from (P1) or (P1') that for all sufficiently large n, the solution $\psi_n(\cdot)$ of (1.1) with initial data ψ_0^n exists at least on $[0,T]$ and $d(\psi_n(t), \psi(t)) \longrightarrow 0$ uniformly on $[0,T]$. Using Corollary 4.13 we infer that for any fixed $t \in [0,T]$ we have $Q(\psi_n(t)) \longrightarrow Q(\psi(t))$. From the first part of the proof and Corollary 2.4 we get $Q(\psi_n(t)) = Q(\psi_0^n) \longrightarrow Q(\psi_0)$ as $n \longrightarrow \infty$. Hence $Q(\psi(t)) = Q(\psi_0)$. $\square$

We now state our orbital stability result, which is based on the argument in [13].

Theorem 7.2 *Assume that (A1), (A2), ((P1) or (P1')) and (P2)–(P4) hold.*

- *We assume $N \geq 2$ and $V \geq 0$ on $[0,\infty)$. Let $q > q_0$, and define $\mathcal{S}_q = \{\psi \in \mathcal{E} \mid Q(\psi) = q,\ \text{and } E(\psi) = E_{min}(q)\}$.*

Then, $\mathcal{S}_q$ is not empty and is orbitally stable by the flow of (1.1) for the semi-distance d_0 in the following sense: for any $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that any solution of (1.1) with initial data ψ_0 such that $d_0(\psi_0, \mathcal{S}_q) < \delta_\varepsilon$ is global and satisfies $d_0(\psi(t), \mathcal{S}_q) < \varepsilon$ for any $t > 0$.

- *Assume that $N = 2$ and* $\inf V < 0$. *Let $q \in (q_0^\sharp, q_\infty^\sharp)$, where $q_0^\sharp, q_\infty^\sharp$ are as in (6.8), and define $\mathcal{S}_q^\sharp = \{\psi \in \mathcal{E} \mid Q(\psi) = q,\ \int_{\mathbf{R}^2} V(|\psi|^2)dx \geq 0 \text{ and } E(\psi) = E_{min}^\sharp(q)\}$.*

Then $\mathcal{S}_q^\sharp$ is orbitally stable by the flow of (1.1) for the semi-distance d_0.

Proof. We argue by contradiction and we assume that the statement is false. Then there is some $\varepsilon_0 > 0$ such that for any $n \geq 1$ there is $\psi_0^n \in \mathcal{E}$ satisfying $d_0(\psi_0^n, \mathcal{S}_q) < \frac{1}{n}$ (resp. $d_0(\psi_0^n, \mathcal{S}_q^\sharp) < \frac{1}{n}$) and there is $t_n > 0$ such that $d_0(\psi_n(t_n), \mathcal{S}_q) \geq \varepsilon_0$ (resp. $d_0(\psi_n(t_n), \mathcal{S}_q^\sharp) \geq \varepsilon_0$), where ψ_n is the solution of the Cauchy problem associated to (1.1) with initial data ψ_0^n.

We claim that $Q(\psi_0^n) \longrightarrow q$ and $E(\psi_0^n) \longrightarrow E_{min}(q)$ (resp. $E(\psi_0^n) \longrightarrow E_{min}^\sharp(q)$). Indeed, for each n there is $\phi_n \in \mathcal{S}_q$ (resp. $\in \mathcal{S}_q^\sharp$) such that $d_0(\psi_0^n, \phi_n) < \frac{2}{n}$. If $N = 2$ and V achieves negative values, we have

$$\limsup_{n\to\infty}\int_{\mathbf{R}^2}|\nabla\psi_0^n|^2\, dx = \limsup_{n\to\infty}\int_{\mathbf{R}^2}|\nabla\phi^n|^2\, dx \leq \limsup_{n\to\infty} E(\phi_n) = E_{min}^\sharp(q) < k_\infty,$$

hence $\int_{\mathbf{R}^2} V(|\psi_0^n|^2)\,dx \geq 0$ for all sufficiently large n. Consider an arbitrary subsequence $(\psi_0^{n_\ell})_{\ell\geq 1}$ of $(\psi_0^n)_{n\geq 1}$. Using either Theorem 4.9 or Theorem 6.2 we infer that there exist a subsequence $(\phi_{n_{\ell_k}})_{k\geq 1}$ of $(\phi_n)_{n\geq 1}$, a sequence $(x_k)_{k\geq 1} \in \mathbf{R}^N$ and $\phi \in \mathcal{S}_q$ (resp. $\in \mathcal{S}_q^\sharp$) such that $d_0(\phi_{n_{\ell_k}}(\cdot + x_k),\,\phi) \longrightarrow 0$ as $k \longrightarrow \infty$. Then $d_0(\psi_0^{n_{\ell_k}}(\cdot + x_k),\,\phi) \leq d_0(\phi_{n_{\ell_k}}(\cdot + x_k),\,\phi) + \frac{2}{n_{\ell_k}} \longrightarrow 0$ and using Corollary 4.13 we get $Q(\psi_0^{n_{\ell_k}}) = Q(\psi_0^{n_{\ell_k}}(\cdot + x_k)) \longrightarrow Q(\phi) = q$ and $E(\psi_0^{n_{\ell_k}}) = E(\psi_0^{n_{\ell_k}}(\cdot + x_k)) \longrightarrow E(\phi) = E_{min}(q)$ (resp. $E(\psi_0^{n_{\ell_k}}) \longrightarrow E(\phi) = E_{min}^\sharp(q)$). Since any subsequence of $(\psi_0^n)_{n\geq 1}$ contains a subsequence as above, the claim follows.

By (P3) and Lemma 7.1 we have $E(\psi_n(t_n)) = E(\psi_0^n) \longrightarrow E_{min}(q)$ (resp. $E(\psi_n(t_n)) \longrightarrow E_{min}^\sharp(q)$) and $Q(\psi_n(t_n)) = Q(\psi_0^n) \longrightarrow q$. Moreover, if $N = 2$ and $\inf V < 0$, we have already seen that $\int_{\mathbf{R}^2} V(|\psi_n(t)|^2)\,dx$ cannot change sign, hence $\int_{\mathbf{R}^2} V(|\psi_n(t_n)|^2)\,dx \geq 0$. Using again either Theorem 4.9 or Theorem 6.2 we see that there are a subsequence $(n_k)_{k\geq 1}$, $y_k \in \mathbf{R}^N$ and $\zeta \in \mathcal{S}_q$ (resp. $\in \mathcal{S}_q^\sharp$) such that $d_0(\phi_{n_k}(t_{n_k}),\,\zeta(\cdot - y_k)) \longrightarrow 0$ as $k \longrightarrow \infty$, and this contradicts the assumption $d_0(\psi_n(t_n), \mathcal{S}_q) \geq \varepsilon_0$ (resp. $d_0(\psi_n(t_n), \mathcal{S}_q^\sharp) \geq \varepsilon_0$) for all n. The proof of Theorem 7.2 is thus complete. $\square$

8 Three families of traveling waves

If the assumptions (A1), (A2) are satisfied and $V \geq 0$ on $[0, \infty)$, Theorem 4.9 and Proposition 4.14 provide finite energy traveling waves to (1.1) with any momentum $q > q_0$; denote by $\mathscr{M}$ the family of these traveling waves. Theorem 5.6 and Proposition 5.8 provide traveling waves that minimize the action $E - cQ$ at constant kinetic energy; let $\mathscr{K}$ be the family of those solutions. If $N = 2$, we have also a family $\mathscr{M}^\sharp$ of traveling waves given by Theorem 6.2. Finally, let $\mathscr{P}$ be the family of traveling waves found in [43]; the elements of $\mathscr{P}$ are minimizers of the action $E - cQ$ under a Pohozaev constraint (see Theorem 8.1 below for a precise statement). Our next goal is to establish relationships between these families of solutions. We will prove that $\mathscr{M} \subset \mathscr{K}$ and $\mathscr{K} \subset \mathscr{P}$ if $N \geq 3$, and that $\mathscr{M} \subset \mathscr{K}$ and $\mathscr{M}^\sharp \subset \mathscr{K}$ if $N = 2$. Besides, we find interesting characterizations of the minima of the associated functionals.

Let

$$(8.1)\quad A(\psi) = \int_{\mathbf{R}^N} \sum_{j=2}^N \left|\frac{\partial\psi}{\partial x_j}\right|^2 dx, \qquad E_c(\psi) = E(\psi) - cQ(\psi), \qquad P_c(\psi) = E_c(\psi) - \frac{2}{N-1}A(\psi).$$

It follows from Proposition 4.1 p. 1091 in [42] that any finite-energy traveling wave ψ of speed c of (1.1) satisfies the Pohozaev identity $P_c(\psi) = 0$. Denote

$$(8.2)\qquad \mathcal{C}_c = \{\psi \in \mathcal{E} \mid \psi \text{ is not constant and } P_c(\psi) = 0\} \quad \text{and} \quad T_c = \inf\{E_c(\psi) \mid \psi \in \mathcal{C}_c\}.$$

We summarize below the main results in [43].

Theorem 8.1 *([43]) Assume that $N \geq 3$ and (A1) and (A2) hold. Then:*

(i) For any $c \in (0, v_s)$ the set $\mathcal{C}_c$ is not empty and $T_c > 0$.

(ii) Let $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ be a sequence such that

$$P_c(\psi_n) \longrightarrow 0 \qquad \text{and} \qquad E_c(\psi_n) \longrightarrow T_c \qquad \text{as } n \longrightarrow \infty.$$

If $N = 3$ we assume in addition that there is a positive constant d such that

$$D(\psi_n) \longrightarrow d \quad \text{as } n \longrightarrow \infty, \qquad \text{where } D(\phi) = \int_{\mathbf{R}^N} \left|\frac{\partial\phi}{\partial x_1}\right|^2 + \frac{1}{2}\left(\varphi^2(|\phi|) - 1\right)^2 dx.$$

Then there exist a subsequence $(\psi_{n_k})_{k\geq 1}$, a sequence $(x_k)_{k\geq 1} \subset \mathbf{R}^N$, and $\psi \in \mathcal{C}_c$ such that $E_c(\psi) = T_c$, that is, ψ is a minimizer of E_c in $\mathcal{C}_c$, $\psi_{n_k}(\cdot + x_k) \longrightarrow \psi$ in $L^p_{loc}(\mathbf{R}^N)$ for $1 \leq p < \infty$ and a.e. on $\mathbf{R}^N$ and

$$\|\nabla \psi_{n_k}(\cdot + x_k) - \nabla \psi\|_{L^2(\mathbf{R}^N)} \longrightarrow 0, \qquad \| \, |\psi_{n_k}|(\cdot + x_k) - |\psi| \, \|_{L^2(\mathbf{R}^N)} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty.$$

(iii) Let ψ be a minimizer of E_c in $\mathcal{C}_c$. Then ψ satisfies (1.3) if $N \geq 4$, respectively there exists $\sigma > 0$ such that $\psi_{1,\sigma}$ satisfies (1.3) if $N = 3$. Moreover, ψ (respectively $\psi_{1,\sigma}$) is a minimum action solution of (1.3), that is it minimizes the action E_c among all finite energy solutions. Conversely, any minimum action solution to (1.3) is a minimizer of E_c in $\mathcal{C}_c$.

Part (i) is Lemma 4.7 in [43], part (ii) follows from Theorems 5.3 and 6.2 there and part (iii) follows from Propositions 5.6 and 6.5 in the same paper and from the fact that any solution ψ satisfies the Pohozaev identity $P_c(\psi) = 0$.

Remark 8.2 As already mentioned in [43] p. 119, all the conclusions of Theorem 8.1 above are valid if $c = 0$ provided that the set $\mathcal{C}_0 = \{\psi \in \mathcal{E} \mid \psi \text{ is not constant and } P_0(\psi) = 0\}$ is not empty. We will see later in section 9 that $\mathcal{C}_0 \neq \emptyset$ if and only if V achieves negative values.

Proposition 8.3 *Assume that $N \geq 3$, (A1) and (A2) hold and $V \geq 0$ on $[0, \infty)$. Then:*
(i) $T_c \geq E_{min}(q) - cq$ for any $q > 0$ and $c \in (0, v_s)$.
(ii) $T_c \longrightarrow \infty$ as $c \longrightarrow 0$.
(iii) Let $\psi \in \mathcal{E}$ be a minimizer of E under the constraint $Q = q_* > 0$. Assume that ψ satisfies an Euler-Lagrange equation $E'(\psi) = cQ'(\psi)$ for some $c \in (0, v_s)$. Then ψ is a minimizer of E_c in $\mathcal{C}_c$.

Proof. For $\psi \in \mathcal{E}$ denote

$$B_c(\psi) = \int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_1}\right|^2 dx - cQ(\psi) + \int_{\mathbf{R}^N} V(|\psi|^2)\, dx. \tag{8.3}$$

Then $E_c(\psi) = A(\psi) + B_c(\psi) = \frac{2}{N-1} A(\psi) + P_c(\psi)$ and $P_c(\psi) = \frac{N-3}{N-1} A(\psi) + B_c(\psi)$.

i) Consider first the case $N \geq 4$. Fix $\psi \in \mathcal{C}_c$. It is clear that $A(\psi) > 0$, hence $B_c(\psi) = P_c(\psi) - \frac{N-3}{N-1} A(\psi) = -\frac{N-3}{N-1} A(\psi) < 0$. Since $V \geq 0$ by hypothesis, it follows that $cQ(\psi) = \int_{\mathbf{R}^N} V(|\psi|^2)\, dx + \int_{\mathbf{R}^N} \left|\frac{\partial \psi}{\partial x_1}\right|^2 dx - B_c(\psi) > 0$, hence $Q(\psi) > 0$ because $c > 0$. It is easy to see that the function $\sigma \longmapsto E_c(\psi_{1,\sigma}) = \sigma^{N-3} A(\psi) + \sigma^{N-1} B_c(\psi)$ achieves its maximum at $\sigma = 1$. Fix $q > 0$. Since $Q(\psi_{1,\sigma}) = \sigma^{N-1} Q(\psi)$, there is $\sigma_q > 0$ such that $Q(\psi_{1,\sigma_q}) = q$. We have obviously $E(\psi_{1,\sigma_q}) \geq E_{min}(q)$ and

$$E_{min}(q) - cq \leq E(\psi_{1,\sigma_q}) - cQ(\psi_{1,\sigma_q}) = E_c(\psi_{1,\sigma_q}) \leq E_c(\psi_{1,1}) = E_c(\psi).$$

Taking the infimum as $\psi \in \mathcal{C}_c$, then the supremum as $q > 0$ in the above inequality we get $\sup\limits_{q>0}(E_{min}(q) - cq) \leq T_c$.

Now consider the case $N = 3$. Let $\psi \in \mathcal{C}_c$. Then $P_c(\psi) = B_c(\psi) = 0$, $Q(\psi) > 0$ and $E_c(\psi_{1,\sigma}) = A(\psi) + \sigma^2 B_c(\psi) = A(\psi)$ for any $\sigma > 0$. Fix $q > 0$. Since $Q(\psi_{1,\sigma}) = \sigma^2 Q(\psi)$, there is $\sigma_q > 0$ such that $Q(\psi_{1,\sigma_q}) = q$ and this implies $E(\psi_{1,\sigma_q}) \geq E_{min}(q)$. We have

$$E_{min}(q) - cq \leq E(\psi_{1,\sigma_q}) - cQ(\psi_{1,\sigma_q}) = E_c(\psi_{1,\sigma_q}) = A(\psi) = E_c(\psi_{1,1}) = E_c(\psi).$$

Since this is true for any $\psi \in \mathcal{C}_c$ and any $q > 0$, we conclude again that $\sup\limits_{q>0}(E_{min}(q) - cq) \leq T_c$.

(ii) Fix $q > \frac{1}{v_s}$. We have $E_{min}(q) - cq > E_{min}(q) - 1$ for any $c \in (0, \frac{1}{q})$. Using (i) we get

$$T_c \geq E_{min}(q) - cq > E_{min}(q) - 1 \qquad \text{for any } c \in (0, \frac{1}{q}).$$

Since $E_{min}(q) \longrightarrow \infty$ as $q \longrightarrow \infty$ by Theorem 4.17 (b), the conclusion follows.

(iii) We know that ψ is a traveling wave of speed c and by Proposition 4.1 p. 1091 in [42] we have $P_c(\psi) = 0$, that is $\psi \in \mathcal{C}_c$. Using (i) we obtain

$$E_c(\psi) \geq T_c \geq \sup_{q>0}(E_{min}(q) - cq). \tag{8.4}$$

On the other hand, we have

$$E_c(\psi) = E(\psi) - cQ(\psi) = E_{min}(q_*) - cq_*.$$

Therefore all inequalities in (8.4) have to be equalities. We infer that ψ minimizes E_c in $\mathcal{C}_c$, $T_c = E_{min}(q_*) - cq_*$ and the function $q \longmapsto E_{min}(q) - cq$ achieves its maximum at q_*. $\square$

The next result shows that the minimizers of E_{min} or $E^\sharp_{min}$ are also minimizers for I_{min} (after scaling).

Proposition 8.4 *Let $N \geq 2$. Assume that (A1), (A2) hold and either*
(a) $V \geq 0$ on $[0, \infty)$ and $q > q_0$, or
(b) $N = 2$, $\inf V < 0$ and $q \in (q_0^\sharp, q_\infty^\sharp)$.
Consider $\psi \in \mathcal{E}$ such that $Q(\psi) = q$ and $E(\psi) = E_{min}(q)$ in case (a), respectively $E(\psi) = E^\sharp_{min}(q)$ in case (b), and ψ satisfies (4.54) for some $c \in (0, v_s)$ (the existence of ψ follows from Theorem 4.9 in case (a) and from Theorem 6.2) in case (b)). Let $k = \int_{\mathbf{R}^N} |\nabla \psi|^2 \, dx$.

Then $\frac{k}{c^{N-2}} > k_0$ and $\psi_{\frac{1}{c}, \frac{1}{c}}$ is a minimizer of I in the set $\{\phi \in \mathcal{E} \mid \int_{\mathbf{R}^2} |\nabla \phi|^2 \, dx = \frac{k}{c^{N-2}}\}$, that is $I(\psi_{\frac{1}{c}, \frac{1}{c}}) = I_{min}\left(\frac{k}{c^{N-2}}\right)$.

Equivalently, ψ is a minimizer of I_c (and of E_c) in the set $\{\phi \in \mathcal{E} \mid \int_{\mathbf{R}^2} |\nabla \phi|^2 \, dx = k\}$).

Moreover, if $N \geq 3$ the function I_{min} is differentiable at $\frac{k}{c^{N-2}}$.

Proof. By Remark 5.9 (i) we have $I_{min}(\frac{k}{v_s^{N-2}}) < -\frac{k}{v_s^N}$ and Proposition 4.14 (i) implies $c \in (0, v_s)$, hence $\frac{k}{c^{N-2}} > \frac{k}{v_s^{N-2}} > k_0$. Using Theorem 5.6 we infer that there is a minimizer $\tilde{\psi} \in \mathcal{E}$ of the functional I under the constraint $\int_{\mathbf{R}^N} |\nabla \tilde{\psi}|^2 \, dx = \frac{k}{c^{N-2}}$. By Proposition 5.8 (ii) there is $c_1 \in (0, v_s)$ such that $\tilde{\psi}_{c_1, c_1}$ satisfies (4.54) with c_1 instead of c.

Let $\psi_1 = \tilde{\psi}_{c,c}$, so that $\int_{\mathbf{R}^N} |\nabla \psi_1|^2 \, dx = c^{N-2} \int_{\mathbf{R}^N} |\nabla \tilde{\psi}|^2 \, dx = k = \int_{\mathbf{R}^N} |\nabla \psi|^2 \, dx$. Denote $q_1 = Q(\psi_1) = c^{N-1} Q(\tilde{\psi})$.

It follows from Proposition 4.1 p. 1091-1092 in [41] that ψ and $\tilde{\psi}_{c_1, c_1}$ satisfy the following Pohozaev identities:

$$-(N-2) \int_{\mathbf{R}^N} |\nabla \psi|^2 \, dx + c(N-1) Q(\psi) = N \int_{\mathbf{R}^N} V(|\psi|^2) \, dx, \tag{8.5}$$

respectively $-(N-2) \int_{\mathbf{R}^N} |\nabla \tilde{\psi}_{c_1, c_1}|^2 \, dx + c_1(N-1) Q(\tilde{\psi}_{c_1, c_1}) = N \int_{\mathbf{R}^N} V(|\tilde{\psi}_{c_1, c_1}|^2) \, dx$. Since $\tilde{\psi}_{c_1, c_1} = (\psi_1)_{\frac{c_1}{c}, \frac{c_1}{c}}$, the latter equality is equivalent to

$$-(N-2) \frac{c_1^{N-2}}{c^{N-2}} \int_{\mathbf{R}^N} |\nabla \psi_1|^2 \, dx + (N-1) \frac{c_1^N}{c^{N-1}} Q(\psi_1) = N \frac{c_1^N}{c^N} \int_{\mathbf{R}^N} V(|\psi_1|^2) \, dx. \tag{8.6}$$

Since $\int_{\mathbf{R}^N} |\nabla \psi_{\frac{1}{c},\frac{1}{c}}|^2\,dx = \frac{k}{c^{N-2}} = \int_{\mathbf{R}^N} |\nabla \tilde{\psi}|^2\,dx$ we have $I(\tilde{\psi}) \le I(\psi_{\frac{1}{c},\frac{1}{c}})$, that is

$$-\frac{1}{c^{N-1}}Q(\psi_1) + \frac{1}{c^N}\int_{\mathbf{R}^N} V(|\psi_1|^2)\,dx \le -\frac{1}{c^{N-1}}Q(\psi) + \frac{1}{c^N}\int_{\mathbf{R}^N} V(|\psi|^2)\,dx. \tag{8.7}$$

Replacing $\int_{\mathbf{R}^N} V(|\psi|^2)\,dx$ and $\int_{\mathbf{R}^N} V(|\psi_1|^2)\,dx$ from (8.5) and (8.6) into (8.7) we get

$$cq + (N-2)k \le cq_1 + (N-2)\frac{c^2}{c_1^2}k. \tag{8.8}$$

Let $\sigma = \left(\frac{q}{q_1}\right)^{\frac{1}{N-1}}$. Then $Q((\psi_1)_{\sigma,\sigma}) = q$ and consequently $E(\psi) \le E((\psi_1)_{\sigma,\sigma})$, that is

$$k + \int_{\mathbf{R}^N} V(|\psi|^2)\,dx \le \sigma^{N-2}k + \sigma^N \int_{\mathbf{R}^N} V(|\psi_1|^2). \tag{8.9}$$

We plug (8.5) and (8.6) into (8.9) to obtain

$$cq_1 + (N-2)\frac{c^2}{c_1^2}k \le Ncq_1 - \frac{N-1}{\sigma^N}cq + \left(\frac{N}{\sigma^2} - \frac{2}{\sigma^N}\right)k. \tag{8.10}$$

Combining (8.10) with (8.8) we infer that $cq + (N-2)k \le Ncq_1 - \frac{N-1}{\sigma^N}cq + \left(\frac{N}{\sigma^2} - \frac{2}{\sigma^N}\right)k$. Since $q = \sigma^{N-1}q_1$, the last inequality can also be written as

$$\frac{cq_1}{\sigma}(\sigma^N - N\sigma + N - 1) + \frac{k}{\sigma^N}((N-2)\sigma^N - N\sigma^{N-2} + 2) \le 0. \tag{8.11}$$

If $N = 2$, (8.11) is equivalent to $\frac{cq_1}{\sigma}(\sigma-1)^2 \le 0$ and it implies that $\sigma = 1$, thus $q = q_1$.

If $N \ge 3$ we have $\sigma^N - N\sigma + N - 1 = (\sigma-1)^2 \sum_{j=0}^{N-2}(N-1-j)\sigma^j$ and $(N-2)\sigma^N - N\sigma^{N-2} + 2 = (\sigma-1)^2\left[(N-2)\sigma^{N-2} + 2\sum_{j=0}^{N-3}(j+1)\sigma^j\right]$. Inserting these identities into (8.11) and using the fact that σ, c, q_1, k are positive we infer that $\sigma = 1$, hence $q = q_1$. Then using (8.8) we obtain $c_1^2 \le c^2$. On the other hand, from (8.10) and the fact that $q = q_1$, $\sigma = 1$ we obtain $c^2 \le c_1^2$. Since c and c_1 are positive, we have necessarily $c = c_1$. Then using (8.5) and (8.6) it is easy to see that $I(\psi_{\frac{1}{c},\frac{1}{c}}) = I(\tilde{\psi})$, hence $I(\psi_{\frac{1}{c},\frac{1}{c}}) = I_{min}(\frac{k}{c^{N-2}})$. Moreover, we have proved that *any* minimizer $\tilde{\psi}$ of I under the constraint $\int_{\mathbf{R}^N} |\nabla\tilde{\psi}|^2\,dx = \frac{k}{c^{N-2}}$ satisfies (5.17) with $\vartheta = -\frac{1}{c^2}$. It follows from Proposition 5.8 (iv) that $d^+I_{min}\left(\frac{k}{c^{N-2}}\right) = d^-I_{min}\left(\frac{k}{c^{N-2}}\right)$, hence I_{min} is differentiable at $\frac{k}{c^{N-2}}$ and $I'_{min}(\frac{k}{c^{N-2}}) = -\frac{1}{c^2}$. $\square$

The next result establishes the relationship, if $N \ge 3$, between the traveling waves obtained from minimizers of I_{min} and the traveling wave solutions given by Theorem 8.1.

Proposition 8.5 *Assume that $N \ge 3$ and (A1), (A2) hold. Let $\mathcal{C}_c$ and T_c be as in (8.2). Then:*

(i) $T_c \ge k + c^N I_{min}\left(\frac{k}{c^{N-2}}\right)$ for any $k > 0$ and any $c \in (0, v_s)$.

(ii) Let ψ be a minimizer of I under the constraint $\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx = k$ and let $c \in (0, v_s)$ be such that $\psi_{c,c}$ satisfies (4.54). Then $\psi_{c,c}$ minimizes $E_c = E - cQ$ in $\mathcal{C}_c$.

Proof. We keep the same notation as in the proof of Proposition 8.3.

(i) Consider the case $N \geq 4$. Fix $\psi \in \mathcal{C}_c$ and $k > 0$. Since $A(\psi) > 0$, the function $\sigma \longmapsto \int_{\mathbf{R}^N} |\nabla \psi_{1,\sigma}|^2 \, dx = \sigma^{N-3} A(\psi) + \sigma^{N-1} \int_{\mathbf{R}^N} |\frac{\partial \psi}{\partial x_1}|^2 \, dx$ is one-to-one from $(0, \infty)$ to $(0, \infty)$, so there is σ_k such that $\int_{\mathbf{R}^N} |\nabla \psi_{1,\sigma_k}|^2 \, dx = k$, that is $\int_{\mathbf{R}^N} |\nabla \psi_{\frac{1}{c}, \frac{\sigma_k}{c}}|^2 \, dx = \frac{k}{c^{N-2}}$. This implies $I\left(\psi_{\frac{1}{c}, \frac{\sigma_k}{c}}\right) \geq I_{min}\left(\frac{k}{c^{N-2}}\right)$. We have $0 = P_c(\psi) = A(\psi) + B_c(\psi)$, thus $A(\psi) > 0 > B_c(\psi)$ and the function $\sigma \longmapsto E_c(\psi_{1,\sigma}) = \sigma^{N-3} A(\psi) + \sigma^{N-1} B_c(\psi)$ achieves its maximum at $\sigma = 1$. Then we have

$$
\begin{aligned}
E_c(\psi) = E_c(\psi_{1,1}) \geq E_c(\psi_{1,\sigma_k}) &= \int_{\mathbf{R}^N} |\nabla \psi_{1,\sigma_k}|^2 \, dx + I_c(\psi_{1,\sigma_k}) \\
&= k + c^N I(\psi_{\frac{1}{c}, \frac{\sigma_k}{c}}) \geq k + c^N I_{min}\left(\frac{k}{c^{N-2}}\right).
\end{aligned}
$$

The above inequality is valid for any $\psi \in \mathcal{C}_c$ and $k > 0$, hence $T_c \geq \sup_{k>0} \left(k + c^N I_{min}\left(\frac{k}{c^{N-2}}\right)\right)$.

Next consider the case $N = 3$. Let $\psi \in \mathcal{C}_c$ and let $k > 0$. Then $P_c(\psi) = B_c(\psi) = 0$ and for any $\sigma > 0$ we have $E_c(\psi_{1,\sigma}) = E_c(\psi) = A(\psi)$ and $\int_{\mathbf{R}^3} |\nabla \psi_{1,\sigma}|^2 \, dx = A(\psi) + \sigma^2 \int_{\mathbf{R}^3} |\frac{\partial \psi}{\partial x_1}|^2 \, dx$. If $A(\psi) \geq k$ we have, taking into account that I_{min} is negative on $(0, \infty)$,

$$
E_c(\psi) = A(\psi) \geq k > k + c^3 I_{min}\left(\frac{k}{c}\right).
$$

If $A(\psi) < k$, there is $\sigma_k > 0$ such that $\int_{\mathbf{R}^3} |\nabla \psi_{1,\sigma_k}|^2 \, dx = k$, which means $\int_{\mathbf{R}^3} |\nabla \psi_{\frac{1}{c}, \frac{\sigma_k}{c}}|^2 \, dx = \frac{k}{c}$. This implies $I_c(\psi_{1,\sigma_k}) = c^3 I\left(\psi_{\frac{1}{c}, \frac{\sigma_k}{c}}\right) \geq c^3 I_{min}\left(\frac{k}{c}\right)$. Thus we get

$$
E_c(\psi) = E_c(\psi_{1,\sigma_k}) = \int_{\mathbf{R}^3} |\nabla \psi_{1,\sigma_k}|^2 \, dx + I_c(\psi_{1,\sigma_k}) \geq k + c^3 I_{min}\left(\frac{k}{c}\right).
$$

Hence $E_c(\psi) \geq k + c^3 I_{min}\left(\frac{k}{c}\right)$ for any $\psi \in \mathcal{C}_c$ and $k > 0$, and the conclusion follows.

(ii) Since $\psi_{c,c}$ satisfies (4.54), by Proposition 4.1 p. 1091 in [42] we have $\psi_{c,c} \in \mathcal{C}_c$. Then

$$
E_c(\psi_{c,c}) \geq T_c \geq \sup_{\kappa>0} \left(\kappa + c^N I_{min}\left(\frac{\kappa}{c^{N-2}}\right)\right). \tag{8.12}
$$

On the other hand,

$$
E_c(\psi_{c,c}) = c^{N-2} \int_{\mathbf{R}^N} |\nabla \psi|^2 \, dx + c^N I(\psi) = c^{N-2} k + c^N I_{min}(k) \leq \sup_{\kappa>0} \left(\kappa + c^N I_{min}\left(\frac{\kappa}{c^{N-2}}\right)\right).
$$

Therefore all inequalities in (8.12) are equalities, $\psi_{c,c}$ minimizes E_c in $\mathcal{C}_c$, $T_c = c^{N-2}k + c^N I_{min}(k)$ and the function $\kappa \longmapsto \kappa + c^N I_{min}\left(\frac{\kappa}{c^{N-2}}\right)$ achieves its maximum at $\kappa = c^{N-2}k$. $\square$

9 Small speed traveling waves

Theorem 4.17 implies that $\frac{E_{min}(q)}{q} \longrightarrow 0$ as $q \longrightarrow \infty$. Since E_{min} is concave and positive, necessarily $d^+ E_{min}(q) \longrightarrow 0$ and $d^- E_{min}(q) \longrightarrow 0$ as $q \longrightarrow \infty$ and we infer that the traveling waves provided by Theorem 4.9 and Proposition 4.14 have speeds close to zero as $q \to \infty$. Similarly, using Lemma 5.3 (i) and (iii) we find that I_{min} is finite for all $k > 0$ and $d^+ I_{min}(k) \longrightarrow -\infty$, $d^- I_{min}(k) \longrightarrow -\infty$ as $k \longrightarrow \infty$ if either $N \geq 3$ or ($N = 2$ and $V \geq 0$). Hence the traveling waves given by Theorem 5.6 and Proposition 5.8 have speeds that tend to zero as $k \longrightarrow \infty$. This section is a first step in understanding the behavior of traveling waves in the limit $c \longrightarrow 0$.

As one would expect, this is related to the existence of finite energy solutions to the stationary version of (1.1), namely to the equation

$$\Delta\psi + F(|\psi|^2)\psi = 0 \qquad \text{in } \mathbf{R}^N. \tag{9.1}$$

Clearly, the solutions of (9.1) are precisely the critical points of E. We call *ground state* of (9.1) a solution that minimizes the energy E among all nontrivial solutions.

Assume that $N \geq 2$ and the assumptions (A1) and (A2) are satisfied. Then (9.1) admits nontrivial solutions $\psi \in \mathcal{E}$ if and only if the nonlinear potential V achieves negative values. The existence follows from Theorem 2.1 p. 100 and Theorem 2.2 p. 103 in [12] if $N \geq 3$, respectively from Theorem 3.1 p. 106 in [12] if $N = 2$. Moreover, the solutions found in [12] are ground states.

On the other hand, any solution $\psi \in \mathcal{E}$ of (9.1) has the regularity provided by Proposition 4.14 (ii) and this is enough to prove that ψ satisfies the Pohozaev identity

$$(N-2)\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx + N\int_{\mathbf{R}^N} V(|\psi|^2)\,dx = 0 \tag{9.2}$$

(see Lemma 2.4 p. 104 in [12]). In particular, (9.2) implies that (9.1) cannot have finite energy solutions if $V \geq 0$.

We will prove in the sequel that if $N \geq 3$ and V achieves negative values, the traveling waves constructed in this paper tend to the ground states of (9.1) as their speed goes to zero. If $N \geq 3$, we have shown in section 8 that all traveling waves found here also belong to the family of traveling waves given by Theorem 8.1, hence it suffices to establish the result for the solutions provided by Theorem 8.1.

If $N = 2$ and V takes negative values, we were not able to prove that $d^{\pm}I_{min}(k) \longrightarrow -\infty$ as $k \longrightarrow k_\infty$. Numerical computations in [17] indicate that this is indeed the case, at least for some model nonlinearities (including the cubic-quintic one). If $\lim\limits_{k\uparrow k_\infty} d^{\pm}I_{min}(k) = -\infty$, the speeds of the traveling waves given by Theorem 5.6 and Proposition 5.8 tend to zero as $k \longrightarrow k_\infty$ and we are able to prove a result similar to Proposition 9.1 below (although the proof is very different because minimization under Pohozaev constraints is no longer possible).

If $V \geq 0$ on $[0,\infty)$, equation (9.1) does not have finite energy solutions. Then the traveling waves of (1.1) have large energy (see Proposition 8.3 (ii)) and are expected to develop vortex structures in the limit $c \longrightarrow 0$. This is the case for the traveling waves to the Gross-Pitaevskii equation: in dimension two the solutions found in [9] have two vortices of opposite sign located at a distance of order $\frac{2}{c}$, and in dimension three the traveling waves found in [8] and [14] have vortex rings. If $V \geq 0$, the behavior of traveling waves in the limit $c \longrightarrow 0$ still needs to be investigated.

Proposition 9.1 *Let $N \geq 3$. Suppose that (A1) and (A2) are satisfied and there exists $s_0 \geq 0$ such that $V(s_0) < 0$. Let $(c_n)_{n\geq 1}$ be any sequence of numbers in $(0, v_s)$ such that $c_n \longrightarrow 0$. For each n, let $\psi_n \in \mathcal{E}$ be any minimizer of $E_{c_n} = E - c_nQ$ in $\mathcal{C}_{c_n}$ such that ψ_n is a traveling wave of (1.1) with speed c_n. Then:*

(i) There are a subsequence $(c_{n_k})_{k\geq 1}$, a sequence $(x_k)_{k\geq 1} \subset \mathbf{R}^N$ and a ground state ψ of (9.1) such that $\psi_{n_k}(\cdot + x_k) \longrightarrow \psi$ in $L^p_{loc}(\mathbf{R}^N)$ for $1 \leq p < \infty$ and a.e. on $\mathbf{R}^N$ and

$$\|\nabla\psi_{n_k}(\cdot + x_k) - \nabla\psi\|_{L^2(\mathbf{R}^N)} \longrightarrow 0, \qquad \|\,|\psi_{n_k}|(\cdot + x_k) - |\psi|\,\|_{L^2(\mathbf{R}^N)} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty.$$

(ii) There is a sequence $(a_k)_{k\geq 1}$ of complex numbers of modulus 1 such that $a_k \longrightarrow 1$ as $k \longrightarrow \infty$ and

$$\|a_k\psi_{n_k}(\cdot + x_k) - \psi\|_{W^{2,p}(\mathbf{R}^N)} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty \quad \text{for any } p \in [2^*, \infty).$$

In particular, $\|a_k\psi_{n_k}(\cdot + x_k) - \psi\|_{C^{1,\alpha}(\mathbf{R}^N)} \longrightarrow 0$ as $k \longrightarrow \infty$ for any $\alpha \in [0,1)$.

If F is C^k it can be proved that the convergence in (ii) holds in $W^{k+2,p}(\mathbf{R}^N)$, $2^* \le p < \infty$.

Proof. (i) Let ψ_0 be any ground state of (9.1). By (9.2) we have $\int_{\mathbf{R}^N} V(|\psi_0|^2)\,dx = -\frac{N-2}{N}\int_{\mathbf{R}^N} |\nabla\psi|^2\,dx < 0$. It is shown in [12] that ψ_0 is a minimizer of the functional $J(\phi) = \int_{\mathbf{R}^N} |\nabla\phi|^2\,dx$ subject to the constraint $\int_{\mathbf{R}^N} V(|\phi|^2)\,dx = \int_{\mathbf{R}^N} V(|\psi_0|^2)\,dx$; conversely, any minimizer of this problem is a ground state to of (9.1), and Proposition 4.14 (ii) implies that any minimizer is C^1 on $\mathbf{R}^N$. It follows from Theorem 2 p. 314 in [42] that any ground state of (9.1) is, after translation, radially symmetric. In particular, the radial symmetry implies that $Q(\psi_0) = 0$.

Let A, $E_c = E - cQ$, P_c be as in (8.1) and $\mathcal{C}_c$ and T_c as in (8.2). Since ψ_0 is a solution of (9.1), it satisfies the Pohozaev identity $P_0(\psi_0) = 0$ and then we get $P_c(\psi_0) = P_0(\psi_0) - cQ(\psi_0) = 0$ for any c, that is $\psi_0 \in \mathcal{C}_c$ for any c. Therefore

(9.3)
$$A(\psi_n) = \frac{N-1}{2}\left(E_{c_n}(\psi_n) - P_{c_n}(\psi_n)\right) = \frac{N-1}{2}E_{c_n}(\psi_n) = \frac{N-1}{2}T_{c_n} \le \frac{N-1}{2}E_{c_n}(\psi_0) = A(\psi_0).$$

On the other hand, by Proposition 10 (ii) in [16] the function $c \longmapsto T_c$ is decreasing on $(0, v_s)$. Fix $c_* \in (0, v_s)$. For all sufficiently large n we have $c_n < c^*$, hence

$$A(\psi_n) = \frac{N-1}{2}T_{c_n} \ge \frac{N-1}{2}T_{c_*} > 0. \tag{9.4}$$

Consider first the case $N \ge 4$. We claim that $E_{GL}(\psi_n)$ is bounded. To see this we argue by contradiction and we assume that there is a subsequence, still denoted $(\psi_n)_{n\ge1}$, such that $E_{GL}(\psi_n) \longrightarrow \infty$. By (9.3) we have

$$D(\psi_n) = \int_{\mathbf{R}^N} \left|\frac{\partial\psi_n}{\partial x_1}\right|^2 + \frac{1}{2}\left(\varphi^2(|\psi_n|) - 1\right)^2\,dx \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty. \tag{9.5}$$

Using Lemma 4.4 (ii) we see that there are two positive constants k_0, ℓ_0 such that for any $\psi \in \mathcal{E}$ satisfying $E_{GL}(\psi) = k_0$ and for any $c \in (0, c_*)$ (where c_* is as in (9.4)) there holds

$$E_c(\psi) \ge E(\psi) - c|Q(\psi)| \ge \ell_0. \tag{9.6}$$

It is easy to see that for each n there is $\sigma_n > 0$ such that

$$E_{GL}((\psi_n)_{\sigma_n,\sigma_n}) = \sigma_n^{N-3}A(\psi_n) + \sigma_n^{N-1}D(\psi_n) = k_0. \tag{9.7}$$

In particular, $(\psi_n)_{\sigma_n,\sigma_n}$ satisfies (9.6).

We recall that the functional B_c was defined in (8.3). We have $B_{c_n}(\psi_n) = P_{c_n}(\psi_n) - \frac{N-3}{N-1}A(\psi_n)$. Then the fact that $P_{c_n}(\psi_n) = 0$ and (9.3) imply that $B_{c_n}(\psi_n)$ is bounded. From (9.5) and (9.7) it follows that $\sigma_n \longrightarrow 0$ as $n \longrightarrow \infty$, hence

$$E_{c_n}((\psi_n)_{\sigma_n,\sigma_n}) = \sigma_n^{N-3}A(\psi_n) + \sigma_n^{N-1}B_{c_n}(\psi_n) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

This contradicts the fact that $E_{c_n}((\psi_n)_{\sigma_n,\sigma_n}) \ge \ell_0$ for all n and the claim is proven.

Using Corollary 4.18 we infer that $Q(\psi_n)$ is bounded. Since $c_n \longrightarrow 0$, using (9.3) we find

$$P_0(\psi_n) = P_{c_n}(\psi_n) + c_nQ(\psi_n) \longrightarrow 0 \qquad \text{and} \tag{9.8}$$

(9.9)
$$\begin{aligned} E(\psi_n) &= E_{c_n}(\psi_n) + c_nQ(\psi_n) = \tfrac{2}{N-1}A(\psi_n) + P_{c_n}(\psi_n) + c_nQ(\psi_n) \\ &\le \tfrac{2}{N-1}A(\psi_0) + c_nQ(\psi_n) = E(\psi_0) + c_nQ(\psi_n) \longrightarrow E(\psi_0) \qquad \text{as } n \longrightarrow \infty. \end{aligned}$$

Then the conclusion follows from Theorem 8.1 (with $c = 0$) and Remark 8.2.

Next consider the case $N = 3$. For all n and all $\sigma > 0$ we have

$$P_{c_n}((\psi_n)_{1,\sigma}) = \sigma^2 P_{c_n}(\psi_n) = 0 \quad \text{and} \quad E_{c_n}((\psi_n)_{1,\sigma}) = A((\psi_n)_{1,\sigma}) + P_{c_n}((\psi_n)_{1,\sigma}) = A(\psi_n) = T_{c_n},$$

hence $(\psi_n)_{1,\sigma}$ is also a minimizer of E_{c_n} in $\mathcal{C}_{c_n}$. For each n there is $\sigma_n > 0$ such that $D((\psi_n)_{1,\sigma_n}) = \sigma_n^2 D(\psi_n) = 1$. We denote $\tilde{\psi}_n = (\psi_n)_{1,\sigma_n}$. Then $\tilde{\psi}_n$ is a minimizer of E_{c_n} in $\mathcal{C}_{c_n}$, $E_{GL}(\tilde{\psi}_n) = A(\tilde{\psi}_n) + 1 = A(\psi_n) + 1$ is bounded by (9.3) and then Corollary 4.18 implies that $Q(\tilde{\psi}_n)$ is bounded. As in the case $N \geq 4$ we find that $(\tilde{\psi}_n)_{n\geq 1}$ satisfies (9.8) and (9.9). From Theorem 8.1 and Remark 8.2 it follows that there exist a subsequence $(\tilde{\psi}_{n_k})_{k\geq 1}$, a sequence $(x_k)_{k\geq 1} \subset \mathbf{R}^3$ and a minimizer $\tilde{\psi}$ of E on $\mathcal{C}_0$ that satisfy the conclusion of Theorem 8.1 (ii). Moreover, there is $\sigma > 0$ such that $\tilde{\psi}$ satisfies the equation

$$\text{(9.10)} \qquad \frac{\partial^2 \tilde{\psi}}{\partial x_1^2} + \sigma^2 \sum_{j=2}^{3} \frac{\partial^2 \tilde{\psi}}{\partial x_j^2} + F(|\tilde{\psi}|^2)\tilde{\psi} = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^3).$$

Let $\psi_k^* = \tilde{\psi}_{n_k}(\cdot + x_k)$. Since ψ_n solves (1.3) with c_n instead of c, it is obvious that ψ_k^* satisfies

$$\text{(9.11)} \qquad ic_{n_k} \frac{\partial \psi_k^*}{\partial x_1} + \frac{\partial^2 \psi_k^*}{\partial x_1^2} + \sigma_{n_k}^2 \sum_{j=2}^{3} \frac{\partial^2 \psi_k^*}{\partial x_j^2} + F(|\psi_k^*|^2)\psi_k^* = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^3).$$

It is easy to see that $\psi_k^* \longrightarrow \tilde{\psi}$ and $F(|\psi_k^*|^2)\psi_k^* \longrightarrow F(|\tilde{\psi}|^2)\tilde{\psi}$ in $\mathcal{D}'(\mathbf{R}^3)$.

We show that $(\sigma_{n_k})_{k\geq 1}$ is bounded. We argue by contradiction and we assume that it contains a subsequence, still denoted the same, that tends to ∞. Multiplying (9.11) by $\frac{1}{\sigma_{n_k}^2}$ and passing to the limit as $k \longrightarrow \infty$ we get

$$\text{(9.12)} \qquad \frac{\partial^2 \tilde{\psi}}{\partial x_2^2} + \frac{\partial^2 \tilde{\psi}}{\partial x_3^2} = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^3).$$

Since $\frac{\partial^2 \tilde{\psi}}{\partial x_j \partial x_k} \in L^p_{loc}(\mathbf{R}^3)$ for any $p \in [1, \infty)$, we infer that the above equality holds in $L^p_{loc}(\mathbf{R}^3)$ for any $p \in [1, \infty)$. By the Sobolev embedding (see Lemma 7 and Remark 4.2 p. 774-775 in [24]) we know that there is $\alpha \in \mathbf{C}$ such that $|\alpha| = 1$ and $\tilde{\psi} - \alpha \in L^6(\mathbf{R}^3)$. Let $\chi \in C_c^\infty(\mathbf{R}^3)$ be a cut-off function such that $\chi = 1$ on $B(0,1)$ and $supp(\chi) \subset B(0,2)$. Taking the scalar product (in $\mathbf{C}$) of (9.12) by $\chi(\frac{x}{n})(\psi - \alpha)$ and letting $n \longrightarrow \infty$ we find $\int_{\mathbf{R}^3} \left|\frac{\partial \tilde{\psi}}{\partial x_2}\right|^2 + \left|\frac{\partial \tilde{\psi}}{\partial x_3}\right|^2 dx = 0$. Since $\psi \in C^{1,\alpha}(\mathbf{R}^3)$, we conclude that $\frac{\partial \tilde{\psi}}{\partial x_2} = \frac{\partial \tilde{\psi}}{\partial x_3} = 0$, hence $\tilde{\psi}$ depends only on x_1. Together with the fact that $\frac{\partial \tilde{\psi}}{\partial x_1} \in L^2(\mathbf{R}^3)$ this implies that $\tilde{\psi}$ is constant, a contradiction. Thus $(\sigma_{n_k})_{k\geq 1}$ is bounded.

If there is a subsequence $(\sigma_{n_{k_j}})_{j\geq 1}$ such that $\sigma_{n_{k_j}} \longrightarrow \sigma_*$ as $j \longrightarrow \infty$, passing to the limit in (9.11) we discover

$$\frac{\partial^2 \tilde{\psi}}{\partial x_1^2} + \sigma_*^2 \sum_{j=2}^{3} \frac{\partial^2 \tilde{\psi}}{\partial x_j^2} + F(|\tilde{\psi}|^2)\tilde{\psi} = 0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^3).$$

If $\sigma_* \neq \sigma$, comparing the above equation to (9.10) we find $\frac{\partial^2 \tilde{\psi}}{\partial x_2^2} + \frac{\partial^2 \tilde{\psi}}{\partial x_3^2} = 0$ in $\mathcal{D}'(\mathbf{R}^3)$ and arguing as previously we infer that $\tilde{\psi}$ is constant, a contradiction. We conclude that necessarily $\sigma_{n_k} \longrightarrow \sigma$ as $k \longrightarrow \infty$. Denoting $\psi = \tilde{\psi}_{1,\frac{1}{\sigma}}$, we easily see that ψ minimizes E in $\mathcal{C}_0$ and is a ground state of (9.1). Then $(\psi_{n_k})_{k\geq 1}$ and ψ satisfy the conclusion of Proposition 9.1 (i).

(ii) By the Sobolev embedding there are α, $\alpha_k \in \mathbf{C}$ of modulus 1 and $C_S > 0$ such that

$$\|\psi_{n_k} - \alpha_k\|_{L^{2^*}(\mathbf{R}^N)} \le C_S \|\nabla \psi_{n_k}\|_{L^2(\mathbf{R}^N)} \qquad \text{and} \qquad \|\psi - \alpha\|_{L^{2^*}(\mathbf{R}^N)} \le C_S \|\nabla \psi\|_{L^2(\mathbf{R}^N)}.$$

We may assume that $\alpha = 1$ for otherwise we multiply ψ_{n_k} and ψ by α^{-1}. (In fact we have $\psi = \alpha \psi_0$, where ψ_0 is real-valued, but we do not need this observation.)

Let $R > 0$ be arbitrary, but fixed. By (i) there exists $k(R) \in \mathbf{N}$ such that for all $k \ge k(R)$ we have $\|\psi_{n_k}(\cdot + x_k) - \psi\|_{L^{2^*}(B(0,R))} < 1$. Then we find

$$\|\alpha_k - 1\|_{L^{2^*}(B(0,R))} \le \|\psi_{n_k}(\cdot + x_k) - \alpha_k\|_{L^{2^*}(\mathbf{R}^N)} + \|\psi_{n_k}(\cdot + x_k) - \psi\|_{L^{2^*}(B(0,R))} + \|\psi - 1\|_{L^{2^*}(\mathbf{R}^N)} \le C$$

for any $k \ge k(R)$, where C does not depend on k. This implies that $\alpha_k \longrightarrow 1$.

Let $\psi_k^* = \alpha_k^{-1} \psi_{n_k}(\cdot + x_k)$, so that $\psi_k^* - \psi \in L^{2^*}(\mathbf{R}^N)$. Using (i) and the Sobolev embedding we get

$$\|\psi_k^* - \psi\|_{L^{2^*}(\mathbf{R}^N)} \le C_S \|\nabla \psi_k^* - \nabla \psi\|_{L^2(\mathbf{R}^N)} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty. \tag{9.13}$$

By (i), $\nabla \psi_k^*$ is bounded in $L^2(\mathbf{R}^N)$ and ψ_k^* is a traveling wave to (1.1) of speed c_{n_k}. It follows from Step 1 in the proof of Lemma 10.1 below that there is $L > 0$, independent of k, such that

$$\|\nabla \psi_k^*\|_{L^\infty(\mathbf{R}^N)} \le L \qquad \text{and} \qquad \|\nabla \psi\|_{L^\infty(\mathbf{R}^N)} \le L.$$

By interpolation we get

$$\|\nabla \psi_k^* - \nabla \psi\|_{L^p(\mathbf{R}^N)} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty \quad \text{for any } p \in [2, \infty). \tag{9.14}$$

Using (9.13), (9.14) and the Sobolev embedding we infer that

$$\|\psi_k^* - \psi\|_{L^p(\mathbf{R}^N)} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty \quad \text{for any } p \in [2^*, \infty]. \tag{9.15}$$

We claim that $\|F(|\psi_k^*|^2)\psi_k^* - F(|\psi|^2)\psi\|_{L^p(\mathbf{R}^N)} \longrightarrow 0$ as $k \longrightarrow \infty$ for any $p \in [2^*, \infty)$. To see this fix $\delta > 0$ such that F is C^1 on $[1-2\delta, 1+2\delta]$ (such δ exists by (A1)). Since $\psi - 1 \in L^{2^*}(\mathbf{R}^N)$ and $\|\nabla \psi\|_{L^\infty(\mathbf{R}^N)} \le L$ we have $\psi \longrightarrow 1$ as $|x| \longrightarrow \infty$, hence there exists $R(\delta) > 0$ verifying $\big||\psi| - 1\big| < \delta$ on $\mathbf{R}^N \setminus B(0, R(\delta))$. By (9.15) there is $k_\delta \in \mathbf{N}$ such that $\|\psi_k^* - \psi\|_{L^\infty(\mathbf{R}^N)} < \delta$ for $k \ge k_\delta$. The mapping $z \longmapsto F(|z|^2)z$ is Lipschitz on $\{z \in \mathbf{C} \mid 1 - 2\delta \le |z| \le 1 + 2\delta\}$, hence there is $C > 0$ such that

$$\big|F(|\psi_k^*|^2)\psi_k^* - F(|\psi|^2)\psi\big| \le C|\psi_k^* - \psi| \qquad \text{on } \mathbf{R}^N \setminus B(0, R(\delta)) \text{ for all } k \ge k_\delta. \tag{9.16}$$

Since $F(|\psi_k^*|^2)\psi_k^* - F(|\psi|^2)\psi$ is bounded and tends a.e. to zero, using Lebesgue's dominated convergence theorem we get

$$\|F(|\psi_k^*|^2)\psi_k^* - F(|\psi|^2)\psi\|_{L^p(B(0,\delta))} \longrightarrow 0 \qquad \text{for any } p \in [1, \infty). \tag{9.17}$$

Now the claim follows from (9.15) - (9.17).

Using the equations satisfied by ψ_k^* and ψ we get

$$\Delta(\psi_k^* - \psi) = -ic_{n_k} \frac{\partial \psi_k^*}{\partial x_1} - \left(F(|\psi_k^*|^2)\psi_k^* - F(|\psi|^2)\psi\right).$$

From the above we infer that $\|\Delta(\psi_k^* - \psi)\|_{L^p(\mathbf{R}^N)} \longrightarrow 0$ for any $p \in [2^*, \infty)$, then using (9.15) and the inequality $\|f\|_{W^{2,p}(\mathbf{R}^N)} \le C_p \Big(\|f\|_{L^p(\mathbf{R}^N)} + \|\Delta f\|_{L^p(\mathbf{R}^N)}\Big)$ we get the desired conclusion. $\square$

10 Small energy traveling waves

The aim of this section is to prove Proposition 1.4. The next lemma shows that the modulus of traveling waves of small energy is close to 1.

Lemma 10.1 *Let $N \geq 2$. Assume that (A1) and ((A2) or (A3)) hold.*

(i) For any $\varepsilon > 0$ there exists $M(\varepsilon) > 0$ such that for any $c \in [0, v_s]$ and for any solution $\psi \in \mathcal{E}$ of (1.3) with $\|\nabla \psi\|_{L^2(\mathbf{R}^N)} < M(\varepsilon)$ we have

$$\big| |\psi(x)| - 1 \big| < \varepsilon \qquad \text{for all } x \in \mathbf{R}^N. \tag{10.1}$$

(ii) Let $p > Np_0$, where p_0 is as in (A2) (respectively $p \geq 1$ if (A3) is satisfied). For any $\varepsilon > 0$ there exists $\ell_p(\varepsilon) > 0$ such that for any $c \in [0, v_s]$ and for any solution $\psi \in \mathcal{E}$ of (1.3) with $\| |\psi| - 1\|_{L^p(\mathbf{R}^N)} < \ell_p(\varepsilon)$, (10.1) holds.

Proof. Assume first that (A1) and (A2) are satisfied.

We will prove that there is $L > 0$ such that any solution $\psi \in \mathcal{E}$ of (1.3) such that $\|\nabla \psi\|_{L^2(\mathbf{R}^N)}$ is sufficiently small (respectively $\||\psi| - r_0\|_{L^2(\mathbf{R}^N)}$ is sufficiently small) satisfies

$$\|\nabla \psi\|_{L^\infty(\mathbf{R}^N)} \leq L. \tag{10.2}$$

Step 1. We prove (10.2) if $N \geq 3$ and $\|\nabla \psi\|_{L^2(\mathbf{R}^N)} \leq M$, where $M > 0$ is fixed.

Using the Sobolev embedding, for any $\phi \in \mathcal{E}$ such that $\|\nabla \phi\|_{L^2(\mathbf{R}^N)} \leq M$ we get

$$\|(|\phi| - 2)_+\|_{L^{2^*}(\mathbf{R}^N)} \leq C_S \|\nabla |\phi|\, \|_{L^2(\mathbf{R}^N)} \leq C_S \|\nabla \phi\|_{L^2(\mathbf{R}^N)}.$$

Since $|\phi| \leq 2 + (|\phi| - 2)_+$, we see that ϕ is bounded in $L^{2^*} + L^\infty(\mathbf{R}^N)$. It follows that for any $R > 0$ there exists $C_{R,M} > 0$ such that for any $\phi \in \mathcal{E}$ as above we have

$$\|\phi\|_{H^1(B(x,R))} \leq C_{R,M} \qquad \text{for all } x \in \mathbf{R}^N.$$

If $c \in [0, v_s]$, $\psi \in \mathcal{E}$ is a solution of (1.3) and $\|\nabla \psi\|_{L^2(\mathbf{R}^N)} \leq M$, using (3.11) and a standard bootstrap argument (which works thanks to (A2)) we infer that for any $p \in [2, \infty)$ there is $\tilde{C}_p > 0$ (depending only on F, N, p and M) such that

$$\|\psi\|_{W^{2,p}(B(x,1))} \leq \tilde{C}_p \qquad \text{for all } x \in \mathbf{R}^N.$$

Then the Sobolev embedding implies that $\psi \in C^{1,\alpha}(\mathbf{R}^N)$ for all $\alpha \in [0, 1)$ and there is $L > 0$ such that (10.2) holds.

Step 2. Proof of (i) in the case $N \geq 3$.

Fix $\varepsilon > 0$. There is $L > 0$ such that any solution $\psi \in \mathcal{E}$ of (1.3) with $\|\nabla \psi\|_{L^2(\mathbf{R}^N)} \leq 1$ satisfies (10.2). If ψ is such a solution and $\big| |\psi(x_0)| - 1 \big| \geq \varepsilon$ for some $x_0 \in \mathbf{R}^N$, from (10.2) we infer that $\big| |\psi(x)| - 1 \big| \geq \frac{\varepsilon}{2}$ for any $x \in B(x_0, \frac{\varepsilon}{2L})$. Then using the Sobolev embedding we get

$$C_S \|\nabla \psi\|_{L^2(\mathbf{R}^N)} \geq \| |\psi| - 1\|_{L^{2^*}(\mathbf{R}^N)} \geq \| |\psi| - 1\|_{L^{2^*}(B(x_0, \frac{\varepsilon}{2L}))} \geq \frac{\varepsilon}{2} \left(\left(\frac{\varepsilon}{2L} \right)^N \mathcal{L}^N(B(0,1)) \right)^{\frac{1}{2^*}}.$$

We conclude that if $\|\nabla \psi\|_{L^2(\mathbf{R}^N)} < \min \left(1, \frac{\varepsilon}{2C_S} \left(\left(\frac{\varepsilon}{2L} \right)^N \mathcal{L}^N(B(0,1)) \right)^{\frac{1}{2^*}} \right)$, then ψ satisfies (10.1).

Step 3. Proof of (10.2) if $N = 2$ and $\|\nabla \psi\|_{L^2(\mathbf{R}^2)}$ is sufficiently small.

By (4.2) there is $M_1 > 0$ such that for any $\phi \in \mathcal{E}$ with $\|\nabla\phi\|_{L^2(\mathbf{R}^2)} \le M_1$ we have

$$\frac{1}{4}\int_{\mathbf{R}^2} \left(\varphi^2(|\phi|)-1\right)^2 dx \le \int_{\mathbf{R}^2} V(|\phi|^2))\, dx \le \frac{3}{4}\int_{\mathbf{R}^2} \left(\varphi^2(|\phi|)-1\right)^2 dx. \tag{10.3}$$

Let $\psi \in \mathcal{E}$ be a solution of (1.3). By Proposition 4.14 (ii) we have $\psi \in W^{2,p}_{loc}(\mathbf{R}^2)$ and this regularity is enough to prove that ψ satisfies the Pohozaev identity

$$-\int_{\mathbf{R}^2} \left|\frac{\partial\psi}{\partial x_1}\right|^2 dx + \int_{\mathbf{R}^2} \left|\frac{\partial\psi}{\partial x_2}\right|^2 dx + \int_{\mathbf{R}^2} V(|\psi|^2))\, dx = 0 \tag{10.4}$$

(see Proposition 4.1 p. 1091 in [41]). In particular, if $\|\nabla\psi\|_{L^2(\mathbf{R}^2)} \le M_1$ by (10.3) and (10.4) we get

$$\int_{\mathbf{R}^2} \left(\varphi^2(|\psi|)-1\right)^2 dx \le 4\int_{\mathbf{R}^2} V(|\psi|^2))\, dx \le 4\int_{\mathbf{R}^2} \left|\frac{\partial\psi}{\partial x_1}\right|^2 dx \le 4M_1 \tag{10.5}$$

and Corollary 4.3 implies that there is some $M_2 > 0$ (independent on ψ) such that $\|\,|\psi|-1\|_{L^2(\mathbf{R}^2)} \le M_2$. We infer that for any $R > 0$ there is $M_3(R) > 0$ (independent on ψ) such that $\|\psi\|_{H^1(B(x,R))} \le M_3(R)$ and hence, by the Sobolev embedding, $\|\psi\|_{L^p(B(x,R))} \le C_p(R)$ for all $x \in \mathbf{R}^2$ and $p \in [2,\infty)$. Using (3.11) and an easy bootstrap argument we get $\|\psi\|_{W^{2,p}(B(x,1))} \le \tilde{C}_p$ for all $x \in \mathbf{R}^2$ and $p \in [1,\infty)$. As in Step 1 we conclude that there is $L > 0$ such that any solution $\psi \in \mathcal{E}$ of (1.3) with $\|\nabla\psi\|_{L^2(\mathbf{R}^2)} \le M_1$ satisfies (10.2).

Step 4. Proof of (i) if $N = 2$.

Fix $\varepsilon > 0$. Let η be as in (3.19) and M_1 as in step 3. If $\psi \in \mathcal{E}$ is a solution of (1.3) with $\|\nabla\psi\|_{L^2(\mathbf{R}^2)} \le M_1$ and there is $x_0 \in \mathbf{R}^2$ such that $\big|\,|\psi(x_0)|-1\big| \ge \varepsilon$, using (10.2) we infer that $\big|\,|\psi(x)|-1\big| \ge \frac{\varepsilon}{2}$ for any $x \in B(x_0, \frac{\varepsilon}{2L})$, hence $\left(\varphi^2(|\psi|)-1\right)^2 \ge \eta(\frac{\varepsilon}{2})$ on $B(x_0, \frac{\varepsilon}{2L})$ and therefore

$$\int_{\mathbf{R}^2} \left(\varphi^2(|\psi|)-1\right)^2 dx \ge \int_{B(x_0,\frac{\varepsilon}{2L})} \left(\varphi^2(|\psi|)-1\right)^2 dx \ge \pi\left(\frac{\varepsilon}{2L}\right)^2 \eta\left(\frac{\varepsilon}{2}\right).$$

On the other hand, by (10.5) we have $\int_{\mathbf{R}^2} \left(\varphi^2(|\psi|)-1\right)^2 dx \le 4\|\nabla\psi\|^2_{L^2(\mathbf{R}^2)}$. We conclude that necessarily $\big|\,|\psi|-1\big| < \varepsilon$ on $\mathbf{R}^2$ if $\|\nabla\psi\|^2_{L^2(\mathbf{R}^2)} < \frac{\pi}{4}\left(\frac{\varepsilon}{2L}\right)^2 \eta(\frac{\varepsilon}{2})$.

Step 5. Proof of (10.2) if $\|\,|\psi|-1\|_{L^p(\mathbf{R}^N)} \le M$ and $p > Np_0$.

By Proposition 4.14 (ii) we know that ψ and $\nabla\psi$ belong to $L^\infty(\mathbf{R}^N)$. We will prove that $\|\psi\|_{L^\infty(\mathbf{R}^N)}$ and $\|\nabla\psi\|_{L^\infty(\mathbf{R}^N)}$ are bounded uniformly with respect to ψ. The constants C_j below depend only on M, F, p, N, but not on ψ.

Let $\phi(x) = e^{\frac{icx_1}{2}}\psi(x)$, so that $|\phi| = |\psi|$ and ϕ satisfies the equation

$$\Delta\phi + \left(\frac{c^2}{4} + F(|\phi|^2)\right)\phi = 0 \qquad \text{in } \mathbf{R}^N. \tag{10.6}$$

For all $x \in \mathbf{R}^N$ we have $\|\phi\|_{L^p(B(x,2))} \le C_1$, where C_1 depends only on M. Fix $r = (\frac{p}{2p_0})^-$ such that $Np_0 < 2rp_0 < p$ and $(2p_0+1)r > p$. In particular, we have $r > \frac{N}{2} \ge 1$. Since $\left|\left(\frac{c^2}{4} + F(|\phi|^2)\right)\phi\right| \le C_2 + C_3|\phi|^{2p_0+1}$, using (10.6) we find that for all $x \in \mathbf{R}^N$ we have

$$\|\Delta\phi\|_{L^r(B(x,2))} \le C_4 + C_5\|\phi\|^{2p_0+1-\frac{p}{r}}_{L^\infty(B(x,2))}\|\phi\|^{\frac{p}{r}}_{L^p(B(x,2))} \le C_6 + C_7\|\phi\|^{2p_0+1-\frac{p}{r}}_{L^\infty(\mathbf{R}^N)}. \tag{10.7}$$

It is obvious that $|\phi| \le C_8 + C_9\|\phi\|^{2p_0+1-\frac{p}{r}}_{L^\infty(\mathbf{R}^N)}|\phi|^{\frac{p}{r}}$, hence ϕ satisfies

$$\|\phi\|_{L^r(B(x,2))} \le (\mathcal{L}^N(B(0,2)))^{\frac{1}{r}}C_8 + C_9C_1^{\frac{p}{r}}\|\phi\|^{2p_0+1-\frac{p}{r}}_{L^\infty(\mathbf{R}^N)}.$$

Then, using (3.11) we infer that for all $x \in \mathbf{R}^N$,

$$\|\phi\|_{W^{2,r}(B(x,1))} \le C_{10} + C_{11}\|\phi\|_{L^\infty(\mathbf{R}^N)}^{2p_0+1-\frac{p}{r}}.$$

Since $r > \frac{N}{2}$, the Sobolev embedding implies $\|\phi\|_{L^\infty(B(x,1))} \le C_s\|\phi\|_{W^{2,r}(B(x,1))}$. Choose $x_0 \in \mathbf{R}^N$ such that $\|\phi\|_{L^\infty(B(x_0,1))} \ge \frac{1}{2}\|\phi\|_{L^\infty(\mathbf{R}^N)}$. We have

$$\frac{1}{2C_s}\|\phi\|_{L^\infty(\mathbf{R}^N)} \le \frac{1}{C_s}\|\phi\|_{L^\infty(B(x_0,1))} \le \|\phi\|_{W^{2,r}(B(x_0,1))} \le C_{10} + C_{11}\|\phi\|_{L^\infty(\mathbf{R}^N)}^{2p_0+1-\frac{p}{r}}.$$

Since $2p_0 + 1 - \frac{p}{r} < 1$ by the choice of r, the above inequality implies that there is $C_{12} > 0$ such that $\|\phi\|_{L^\infty(\mathbf{R}^N)} \le C_{12}$. Then using (10.6) and (3.11) we infer that $\|\phi\|_{W^{2,q}(B(x,1))} \le C(q)$ for all $x \in (\mathbf{R}^N)$ and all $q \in (1,\infty)$, and the Sobolev embedding implies $\|\nabla\phi\|_{L^\infty(\mathbf{R}^N)} \le C_{13}$ for some $C_{13} > 0$. Since $\psi(x) = e^{-\frac{icx_1}{2}}\phi(x)$, the conclusion follows.

Step 6. Proof of (ii).

Let ψ be a solution of (1.3) such that $\|\,|\psi| - 1\|_{L^p(\mathbf{R}^N)} \le 1$. By step 5, there is $L > 0$ (independent on ψ) such that (10.2) holds. If there is $x_0 \in \mathbf{R}^N$ such that $\big|\,|\psi(x_0)| - 1\big| \ge \varepsilon$, we have $\big|\,|\psi| - 1\big| \ge \frac{\varepsilon}{2}$ on $B(x_0, \frac{\varepsilon}{2L})$ and consequently

$$\|\,|\psi| - 1\|_{L^p(\mathbf{R}^N)} \ge \|\,|\psi| - 1\|_{L^p(B(x_0,\frac{\varepsilon}{2L}))} \ge \frac{\varepsilon}{2}\left(\left(\frac{\varepsilon}{2L}\right)^N \mathcal{L}^N(B(0,1))\right)^{\frac{1}{p}}.$$

Thus necessarily $\big|\,|\psi(x)|-1\big| < \varepsilon$ on $\mathbf{R}^N$ if $\|\,|\psi|-1\|_{L^p(\mathbf{R}^N)} < \min\left(1, \frac{\varepsilon}{2}\left(\left(\frac{\varepsilon}{2L}\right)^N \mathcal{L}^N(B(0,1))\right)^{\frac{1}{p}}\right)$.

If (A1) and (A3) hold, it follows from the proof of Proposition 2.2 (i) p. 1078 in [41] that there is $L > 0$ such that (10.2) holds for any $c \in [0, v_s]$ and any solution $\psi \in \mathcal{E}$ of (1.3). Therefore the conclusions of steps 1, 3 and 5 are automatically satisfied. The rest of the proof is exactly as above. $\square$

By (A1) we may fix $\beta_* > 0$ such that $\frac{1}{4}(s-1)^2 \le V(s) \le \frac{3}{4}(s-1)^2$ if $|\sqrt{s} - 1| \le \beta_*$.

Let $U \in \mathcal{E}$ be a traveling wave to (1.1) such that $1 - \beta_* \le |U| \le 1 + \beta_*$. It is clear that

$$\frac{1}{4}(|U|^2 - 1)^2 \le V(|U|^2) \le \frac{3}{4}(|U|^2 - 1)^2 \qquad \text{on } \mathbf{R}^N. \tag{10.8}$$

It is an easy consequence of Theorem 3 p. 38 and of Lemma C1 p. 66 in [11] that there exists a lifting $U = \rho e^{i\theta}$ on $\mathbf{R}^N$, where $\rho, \theta \in W^{2,p}_{loc}(\mathbf{R}^N)$ for any $p \in [1,\infty)$. Then (1.3) can be written in the form

$$\begin{cases} \Delta\rho - \rho|\nabla\theta|^2 + \rho F(\rho^2) = c\rho\dfrac{\partial\theta}{\partial x_1}, \\[2ex] \operatorname{div}(\rho^2\nabla\theta) = -\dfrac{c}{2}\dfrac{\partial}{\partial x_1}(\rho^2 - 1). \end{cases} \tag{10.9}$$

Multiplying the first equation in (10.9) by ρ we get

$$\frac{1}{2}\Delta(\rho^2 - 1) - |\nabla U|^2 + \rho^2 F(\rho^2) - c(\rho^2 - 1)\frac{\partial\theta}{\partial x_1} = c\frac{\partial\theta}{\partial x_1}. \tag{10.10}$$

The second equation in (10.9) can be written as

$$\operatorname{div}((\rho^2 - 1)\nabla\theta) + \frac{c}{2}\frac{\partial}{\partial x_1}(\rho^2 - 1) = -\Delta\theta. \tag{10.11}$$

We set $\eta = \rho^2 - 1$ and define $g : [-1, +\infty)$ by $g(s) = v_s^2 s + 2(1+s)F(1+s)$, so that $g(s) = \mathcal{O}(s^2)$ for $s \longrightarrow 0$. Taking the Laplacian of (10.10) and applying the operator $c\frac{\partial}{\partial x_1}$ to (10.11), then summing up the resulting equalities we find

(10.12)
$$\left[\Delta^2 - v_s^2\Delta + c^2\partial_{x_1}^2\right]\eta = \Delta\left(2|\nabla U|^2 - g(\eta) + 2c\eta\partial_{x_1}\theta\right) - 2c\partial_{x_1}(\mathrm{div}(\eta\nabla\theta)) \quad \text{in } \mathcal{S}'(\mathbf{R}^N).$$

Notice that the right-hand side of (10.12) contains terms that are (at least) quadratic. We write (10.12) using the Fourier transform as

$$\hat{\eta}(\xi) = \mathcal{L}_c(\xi)\hat{\Upsilon}(\xi), \tag{10.13}$$

where

$$\widehat{\Upsilon}(\xi) = -\mathcal{F}(2|\nabla U|^2 - g(\eta)) - 2c\frac{|\xi|^2 - \xi_1^2}{|\xi|^2}\mathcal{F}(\eta\partial_{x_1}\phi) + 2c\sum_{j=2}^{N}\frac{\xi_1\xi_j}{|\xi|^2}\mathcal{F}(\eta\partial_{x_j}\phi) \tag{10.14}$$

and

$$\mathcal{L}_c(\xi) = \frac{|\xi|^2}{|\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2}. \tag{10.15}$$

On the other hand, we know that U satisfies the Pohozaev identity (8.5). Using (2.7) and the Cauchy-Schwarz identity we have

$$|Q(U)| = \left|\int_{\mathbf{R}^N}(\rho^2 - 1)\theta_{x_1}\,dx\right| \le \|\eta\|_{L^2(\mathbf{R}^N)}\|\theta_{x_1}\|_{L^2(\mathbf{R}^N)} \le \frac{1}{1-\beta_*}\|\eta\|_{L^2(\mathbf{R}^N)}\|\nabla U\|_{L^2(\mathbf{R}^N)}.$$

Inserting this estimate into (8.5), using (10.8) and the fact that $|c| \le v_s$ we get

$$(N-2)\|\nabla U\|_{L^2(\mathbf{R}^N)}^2 - \frac{(N-1)v_s}{1-\beta_*}\|\eta\|_{L^2(\mathbf{R}^N)}\|\nabla U\|_{L^2(\mathbf{R}^N)} + \frac{N}{4}\|\eta\|_{L^2(\mathbf{R}^N)}^2 \le 0. \tag{10.16}$$

The case $N \ge 3$. If $N \ge 3$, let $a_1 \le a_2$ be the two roots of the equation $(N-2)y^2 - \frac{(N-1)v_s}{1-\beta_*}y + \frac{N}{4} = 0$. It is obvious that a_1 and a_2 are positive and from (10.16) we infer that

$$a_1\|\eta\|_{L^2(\mathbf{R}^N)} \le \|\nabla U\|_{L^2(\mathbf{R}^N)} \le a_2\|\eta\|_{L^2(\mathbf{R}^N)}. \tag{10.17}$$

Proof of Proposition 1.4 for $N \ge 3$. We use the ideas introduced in [7] and [21].

In the following C_j and K_j are positive constants depending only on N and F.

Let β_* be as above. By Lemma 10.1, there are $M_1, \ell_1 > 0$ such that any solution $U \in \mathcal{E}$ to (1.3) with $\int_{\mathbf{R}^N}|\nabla U|^2\,dx \le M_1$ (respectively with $\int_{\mathbf{R}^N}\left(|U|^2 - 1\right)^2\,dx \le \ell_1$ if (A3) holds or if (A2) holds and $p_0 < \frac{2}{N}$) satisfies $1 - \beta_* \le |U| \le 1 + \beta_*$ and, in addition, (10.2) is verified. Then we have a lifting $U = \rho e^{i\theta}$ and (10.8)-(10.17) hold. Since $g(\eta) = \mathcal{O}(\eta^2)$, it follows from (10.17) that

$$\begin{aligned}\|2|\nabla U|^2 - g(\eta)\|_{L^1(\mathbf{R}^N)} &\le 2\|\nabla U\|_{L^2(\mathbf{R}^N)}^2 + C_1\|\eta\|_{L^2(\mathbf{R}^N)}^2 \le C_2\|\eta\|_{L^2(\mathbf{R}^N)}^2 \\ &\le C_3\|\nabla U\|_{L^2(\mathbf{R}^N)}^2.\end{aligned} \tag{10.18}$$

On the other hand, from $1 - \beta_* \le |U| \le 1 + \beta_*$ and (10.2) we get $\|2|\nabla U|^2 - g(\eta)\|_{L^\infty(\mathbf{R}^N)} \le C_4$ and then, by interpolation,

$$\|2|\nabla U|^2 - g(\eta)\|_{L^p(\mathbf{R}^N)} \le C_1(p)\|\eta\|_{L^2(\mathbf{R}^N)}^{\frac{2}{p}}, \tag{10.19}$$

respectively

$$\|2|\nabla U|^2 - g(\eta)\|_{L^p(\mathbf{R}^N)} \leq K_1(p)\|\nabla U\|_{L^2(\mathbf{R}^N)}^{\frac{2}{p}} \tag{10.20}$$

for any $p \in [1, \infty)$. It is obvious that $|\eta \partial_{x_j}\theta| \leq \frac{1}{1-\beta_*}|\eta| \cdot |\nabla U|$ and, as above, we find

$$\|\eta \partial_{x_j}\theta\|_{L^p(\mathbf{R}^N)} \leq C_2(p)\|\eta\|_{L^2(\mathbf{R}^N)}^{\frac{2}{p}}, \quad \text{and} \quad \|\eta \partial_{x_j}\theta\|_{L^p(\mathbf{R}^N)} \leq K_2(p)\|\nabla U\|_{L^2(\mathbf{R}^N)}^{\frac{2}{p}}. \tag{10.21}$$

By the standard theory of Riesz operators (see, e.g., [45]), the functions $\xi \longmapsto \frac{\xi_j \xi_k}{|\xi|^2}$ are Fourier multipliers from $L^p(\mathbf{R}^N)$ to $L^p(\mathbf{R}^N)$, $1 < p < \infty$. Using (10.14) and (10.19)-(10.21) we infer that $\Upsilon \in L^p(\mathbf{R}^N)$ for $1 < p < \infty$ and

$$\|\Upsilon\|_{L^p(\mathbf{R}^N)} \leq C_3(p)\|\eta\|_{L^2(\mathbf{R}^N)}^{\frac{2}{p}}, \quad \text{respectively} \quad \|\Upsilon\|_{L^p(\mathbf{R}^N)} \leq K_3(p)\|\nabla U\|_{L^2(\mathbf{R}^N)}^{\frac{2}{p}}. \tag{10.22}$$

We will use the following result, which is Lemma 3.3 p. 377 in [21] with $\alpha = \frac{2}{2N-1}$ and $q = 2$. Notice that $\frac{1}{1-\alpha} = \frac{2N-1}{2N-3} < 2$ if $N \geq 3$.

Lemma 10.2 *([21]) Let $N \geq 3$ and let $p_N = \frac{2(2N-1)}{2N+3} \in (1, 2)$. There exists a constant K_N, depending only on N, such that for any $c \in [0, v_s]$ and any $f \in L^{p_N}(\mathbf{R}^N)$ we have*

$$\|\mathcal{F}^{-1}(\mathcal{L}_c(\xi)\mathcal{F}(f))\|_{L^2(\mathbf{R}^N)} \leq K_N\|f\|_{L^{p_N}(\mathbf{R}^N)}.$$

From (10.13), Lemma 10.2 and (10.22) we get

$$\|\eta\|_{L^2(\mathbf{R}^N)} \leq K_N\|\Upsilon\|_{L^{p_N}(\mathbf{R}^N)} \leq K_N C_3(p_N)\|\eta\|_{L^2(\mathbf{R}^N)}^{\frac{2}{p_N}}. \tag{10.23}$$

Since $\frac{2}{p_N} > 1$, (10.23) implies that there is $\ell_* > 0$ (depending only on N and F) such that $\|\eta\|_{L^2(\mathbf{R}^N)} \geq \ell_*$, or $\|\eta\|_{L^2(\mathbf{R}^N)} = 0$. In the latter case from (10.17) we get $\|\nabla U\|_{L^2(\mathbf{R}^N)} = 0$, hence U is constant.

From (10.23) and (10.17) we obtain

$$\|\nabla U\|_{L^2(\mathbf{R}^N)} \leq a_2\|\eta\|_{L^2(\mathbf{R}^N)} \leq a_2 K_N C_3(p_N)\|\eta\|_{L^2(\mathbf{R}^N)}^{\frac{2}{p_N}} \leq a_1^{-\frac{2}{p_N}} a_2 K_N C_3(p_N)\|\nabla U\|_{L^2(\mathbf{R}^N)}^{\frac{2}{p_N}}.$$

As above we infer that there is $k_* > 0$ such that either $\|\nabla U\|_{L^2(\mathbf{R}^N)} \geq k_*$, or U is constant. $\square$

The case $N = 2$. If $N = 2$, from (10.16) we infer that $\|\eta\|_{L^2(\mathbf{R}^N)} \leq \frac{2v_s}{1-\beta_*}\|\nabla U\|_{L^2(\mathbf{R}^N)}$. However, the Pohozaev identities alone do not imply an estimate of the form $\|\nabla U\|_{L^2(\mathbf{R}^N)} \leq C\|\eta\|_{L^2(\mathbf{R}^N)}$. To prove this we need the following two identities, which are valid in any space dimension and are of independent interest.

Lemma 10.3 *Let $U = \rho e^{i\theta} \in \mathcal{E}$ be a solution of (1.3), where* $\inf \rho > 0$ *and ρ is bounded. Then we have*

$$2\int_{\mathbf{R}^N} \rho^2|\nabla\theta|^2 \, dx = -c\int_{\mathbf{R}^N} (\rho^2 - 1)\partial_{x_1}\theta \, dx \qquad \textit{and} \tag{10.24}$$

$$\int_{\mathbf{R}^N} 2\rho|\nabla\rho|^2 + \rho(\rho^2 - 1)|\nabla\theta|^2 - \rho(\rho^2 - 1)F(\rho^2) \, dx = -c\int_{\mathbf{R}^N} \rho(\rho^2 - 1)\partial_{x_1}\theta \, dx. \tag{10.25}$$

Proof. Formally, U is a critical point of the functional $E_c = E - cQ$. Denoting $U(s) = \rho e^{is\theta}$ one would expect that $\frac{d}{ds}_{|s=1}(E_c(U(s)) = 0$ and this is precisely (10.24).

In the case of the Gross-Pitaevskii equation, (10.24) was proven in [7] (see Lemma 2.8 p. 594 there) by multiplying the second equation in (10.9) by θ, then integrating by parts. The integrations are justified by the particular decay at infinity of traveling waves for the Gross-Pitaevskii equation. Since such decay properties have not been rigorously established for other nonlinearities, we proceed as follows.

For $R > 0$, we denote $\bar{\theta} = \frac{1}{\mathcal{H}^{N-1}(\partial B(0,R))} \int_{\partial B_R} \theta \, d\mathcal{H}^{N-1}$, we multiply the second equation in (10.9) by $\theta - \bar{\theta}$ and integrate by parts over $B(0, R)$. We get

$$\begin{aligned} &2\int_{B(0,R)} \rho^2 |\nabla\theta|^2 \, dx - 2\int_{\partial B(0,R)} \rho^2 \frac{\partial\theta}{\partial\nu}(\theta - \bar{\theta}) \, d\mathcal{H}^{N-1} \\ &= -c\int_{B(0,R)} (\rho^2 - 1)\partial_{x_1}\theta \, dx + c\int_{\partial B(0,R)} (\rho^2 - 1)(\theta - \bar{\theta})\nu_1 \, d\mathcal{H}^{N-1}, \end{aligned} \tag{10.26}$$

where ν is the outward unit normal to $\partial B(0, R)$. By the Poincaré inequality we have for some constant C independent of R,

$$\|\theta - \bar{\theta}\|_{L^2(\partial B(0,R))} \leq CR\|\nabla\theta\|_{L^2(\partial B(0,R))}.$$

Using the boundedness of ρ and the Cauchy-Schwarz inequality we have for $R \geq 1$

$$\begin{aligned} &\left|2\int_{\partial B(0,R)} \rho^2(\theta - \bar{\theta})\frac{\partial\theta}{\partial\nu} \, d\mathcal{H}^{N-1}\right| + \left|c\int_{\partial B(0,R)} (\rho^2 - 1)(\theta - \bar{\theta})\nu_1 \, d\mathcal{H}^{N-1}\right| \\ &\qquad\qquad \leq CR\int_{\partial B(0,R)} (\rho^2 - 1)^2 + |\nabla\theta|^2 \, d\mathcal{H}^{N-1}. \end{aligned}$$

Since $\rho^2 - 1 \in L^2(\mathbf{R}^N)$ and $\nabla\theta \in L^2(\mathbf{R}^N)$, we have

$$\int_1^{+\infty} \left(\int_{\partial B(0,R)} (\rho^2 - 1)^2 + |\nabla\theta|^2 \, d\mathcal{H}^{N-1}\right) dR = \int_{\{|x|\geq 1\}} (\rho^2 - 1)^2 + |\nabla\theta|^2 \, dx < \infty,$$

hence there exists a sequence $R_j \longrightarrow +\infty$ such that

$$\int_{\partial B(0,R_j)} (\rho^2 - 1)^2 + |\nabla\theta|^2 \, d\mathcal{H}^{N-1} \leq \frac{1}{R_j \ln R_j}.$$

Writing (10.26) for each j, then passing to the limit as $j \longrightarrow \infty$ we obtain (10.24).

It is easily seen that $\rho^2 - 1 \in H^1(\mathbf{R}^N)$. Multiplying the first equation in (10.9) by $\rho^2 - 1$ and using the standard integration by parts formula for H^1 functions (cf. [10] p. 197) we get (10.25). □

Using (10.24) and the Cauchy-Schwarz inequality we get

$$2\int_{\mathbf{R}^N} \rho^2|\nabla\theta|^2 \, dx = -c\int_{\mathbf{R}^N} (\rho^2 - 1)\partial_{x_1}\theta \, dx \leq C\left(\int_{\mathbf{R}^N} (\rho^2 - 1)^2 \, dx\right)^{1/2} \left(\int_{\mathbf{R}^N} \rho^2|\nabla\theta|^2 \, dx\right)^{1/2},$$

from which it comes

$$\int_{\mathbf{R}^N} \rho^2|\nabla\theta|^2 \, dx \leq C\int_{\mathbf{R}^N} \eta^2 \, dx.$$

Using (10.25), the fact that $0 < 1 - \beta_* \le \rho \le 1 + \beta_*$, the inequality $2ab \le a^2 + b^2$ and the above estimate we find

$$\begin{aligned} 2(1-\beta_*) \int_{\mathbf{R}^N} |\nabla \rho|^2 \, dx \le & \int_{\mathbf{R}^N} 2\rho |\nabla \rho|^2 \, dx \\ &= - \int_{\mathbf{R}^N} \rho\eta |\nabla\theta|^2 \, dx - c\int_{\mathbf{R}^N} \eta\rho\partial_{x_1}\theta \, dx + \int_{\mathbf{R}^N} \rho^2 \eta F(\rho^2) \, dx \\ &\le C \int_{\mathbf{R}^N} \rho^2|\nabla\theta|^2 + \eta^2 \, dx \le C\int_{\mathbf{R}^N} \eta^2 \, dx. \end{aligned}$$

It follows from the above inequalities that in the case $N = 2$, there exist two positive constants a_1, a_2 such that any solution $U \in \mathcal{E}$ to (1.3) with $0 \le c \le v_s$ and $1-\beta_* \le |U| \le 1+\beta_*$ satisfies (10.17).

Proof of Proposition 1.4 if $N = 2$, (A4) holds and $F''(1) = 3$. The strategy used in the case $N \ge 3$ has to be adapted: small energy traveling waves do exist when $N = 2$ and $F''(1) \ne 3$ (see Theorem 4.9, Proposition 4.14 and Theorem 4.15). This is related to the fact that Lemma 10.2 does not apply if $N = 2$. The proof relies on an expansion in the small parameter η and the observation that when the energy is small, we must have $\partial_{x_1}\phi \simeq -c\eta/2$. Since $v_s^2 = 2 = -2F'(1)$ and $F''(1) = 3$, by (A4) the function g has the expansion as $s \longrightarrow 0$

$$\begin{aligned} g(s) &= v_s^2 s + 2(1+s)F(1+s) = v_s^2 s + 2(1+s)\Big(sF'(1) + \frac{1}{2}s^2F''(1) + \mathcal{O}(s^3)\Big) \\ &= s^2 + \mathcal{O}(s^3). \end{aligned}$$

By Lemma 10.1, there are M_1, $\ell_1 > 0$ such that any solution $U \in \mathcal{E}$ to (1.3) with $c \in [0, v_s]$ and $\int_{\mathbf{R}^2} |\nabla U|^2 \, dx \le M_1$ (respectively $\int_{\mathbf{R}^2} \big(|U|^2 - 1\big)^2 \, dx \le \ell_1$ if (A3) holds or if (A2) holds and $p_0 < 1$) satisfies $1 - \beta_* \le |U| \le 1 + \beta_*$, the estimate (10.2) is verified, we have a lifting $U = \rho e^{i\theta}$ and all the statements above are valid.

Recalling that Υ is defined by (10.14), we observe that in the expression of $2|\nabla U|^2 - g(\eta)$ we have the almost cancellation of two quadratic terms: $2\rho^2(\partial_{x_1}\theta)^2 - \eta^2 \simeq 2((\partial_{x_1}\theta)^2 - \frac{v_s^2}{4}\eta^2)$ is much smaller than quadratic if $\partial_{x_1}\theta \simeq v_s\eta/2$. We now quantify this idea and split the proof into 7 steps. We denote

$$h = \partial_{x_1}\theta + \frac{c}{2}\eta. \tag{10.27}$$

By Lemma 10.1, $\|\eta\|_{L^\infty(\mathbf{R}^2)}$ can be made arbitrarily small by taking M_1 (respectively ℓ_1) sufficiently small. Moreover, using (10.17) we get

$$\|\eta\|^4_{L^4(\mathbf{R}^2)} \le \|\eta\|^2_{L^\infty(\mathbf{R}^2)}\|\eta\|^2_{L^2(\mathbf{R}^2)} \le C\|\eta\|^2_{L^2(\mathbf{R}^2)} \le C\|\nabla U\|^2_{L^2(\mathbf{R}^2)}. \tag{10.28}$$

Step 1. There is $C > 0$, depending only on F, such that if M_1 (respectively ℓ_1) is small enough,

$$\int_{\mathbf{R}^2} h^2 + (\partial_{x_2}\theta)^2 + (v_s^2 - c^2)\eta^2 \, dx \le C\|\eta\|^4_{L^4(\mathbf{R}^2)}.$$

The starting point is the integral identity

$$\int_{\mathbf{R}^2} \rho^2|\nabla\theta|^2 + V(\rho^2) + c(\rho^2 - 1)\partial_{x_1}\theta \, dx = 0,$$

which comes from the combination of (10.24) and the Pohozaev identity $\int_{\mathbf{R}^2} 2V(\rho^2) + c(\rho^2 - 1)\partial_{x_1}\theta \, dx = 0$ (see Proposition 4.1 in [41]). From (A4) with $F''(1) = 3$ we have the Taylor expansion of the potential

$$V(\rho^2) = V(1+\eta) = \frac{v_s^2}{4}\eta^2 - \frac{1}{6}F''(1)\eta^3 + \mathcal{O}(\eta^4) = \frac{v_s^2}{4}\eta^2 - \frac{v_s^2}{4}\eta^3 + \mathcal{O}(\eta^4).$$

Therefore, the above integral identity gives

$$\int_{\mathbf{R}^2} (1+\eta)(\partial_{x_2}\theta)^2 + (\partial_{x_1}\theta)^2 + \eta(\partial_{x_1}\theta)^2 + \frac{v_s^2}{4}\eta^2 - \frac{v_s^2}{4}\eta^3 + \mathcal{O}(\eta^4) + c\eta\partial_{x_1}\theta \; dx = 0.$$

Then the identity $h^2 = (\partial_{x_1}\theta)^2 + c\eta\partial_{x_1}\theta + \frac{c^2}{4}\eta^2$ gives

$$\int_{\mathbf{R}^2} (1+\eta)(\partial_{x_2}\theta)^2 + h^2 + \frac{v_s^2 - c^2}{4}\eta^2 + \eta(\partial_{x_1}\theta)^2 - \frac{v_s^2}{4}\eta^3 \; dx = -\int_{\mathbf{R}^2} \mathcal{O}(\eta^4) \; dx,$$

hence, rearranging the cubic terms,

$$\int_{\mathbf{R}^2} (1+\eta)(\partial_{x_2}\theta)^2 + h^2 + \frac{v_s^2 - c^2}{4}\eta^2 \left(1-\eta\right) \; dx = -\int_{\mathbf{R}^2} \eta h \left(h - c\eta\right) + \mathcal{O}(\eta^4) \; dx. \tag{10.29}$$

For the left-hand side, we have $1+\eta \geq \frac{1}{2}$ and $1-\eta \geq \frac{1}{2}$ if M_1 or ℓ_1 are sufficiently small (because $\|\eta\|_{L^\infty(\mathbf{R}^2)}$ is small). We now estimate the right-hand side. Since $\|\eta\|_{L^\infty(\mathbf{R}^2)}$ is small, we have $|\int_{\mathbf{R}^2} \mathcal{O}(\eta^4) \; dx| \leq C\|\eta\|^4_{L^4(\mathbf{R}^2)}$ and by Cauchy-Schwarz and the inequality $2ab \leq a^2 + b^2$,

$$\begin{aligned} \Big| \int_{\mathbf{R}^2} \eta h \left(h - c\eta\right) \; dx \Big| &\leq \|\eta\|_{L^\infty(\mathbf{R}^2)} \int_{\mathbf{R}^2} h^2 \; dx + c\Big(\int_{\mathbf{R}^2} h^2 \; dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbf{R}^2} \eta^4 \; dx\Big)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \int_{\mathbf{R}^2} h^2 \; dx + C\|\eta\|^4_{L^4(\mathbf{R}^2)}, \end{aligned}$$

provided that M_1 or ℓ_1 are small enough, where C depends only on F. Inserting these estimates into (10.29) yields the result.

Step 2. There exists C, depending only on F, such that for $M1$ (respectively ℓ_1) small enough,

$$\int_{\mathbf{R}^2} |\nabla\rho|^2 + (v_s^2 - c^2)\eta^2 \; dx \leq C\|\eta\|^2_{L^4(\mathbf{R}^2)}.$$

We start from (10.25), that we write in the form

$$\int_{\mathbf{R}^2} 2\rho|\nabla\rho|^2 \; dx = -\int_{\mathbf{R}^2} \rho\eta \left((\partial_{x_1}\theta)^2 + (\partial_{x_2}\theta)^2\right) - \rho\eta F(\rho^2) + c\rho\eta\partial_{x_1}\theta \; dx.$$

Using the expansion $F(\rho^2) = \eta F'(1) + \mathcal{O}(\eta^2) = -\frac{v_s^2\eta}{2} + \mathcal{O}(\eta^2)$, this gives

$$\int_{\mathbf{R}^2} 2\rho|\nabla\rho|^2 + \frac{v_s^2 - c^2}{2}\rho\eta^2 \; dx = -\int_{\mathbf{R}^2} \rho\eta \left((\partial_{x_1}\theta)^2 + (\partial_{x_2}\theta)^2\right) + c\rho\eta h + \mathcal{O}(|\eta|^3) \; dx. \tag{10.30}$$

Note that by the Cauchy-Schwarz inequality,

$$\|\eta\|^3_{L^3(\mathbf{R}^2)} \leq \|\eta\|^2_{L^4(\mathbf{R}^2)}\|\eta\|_{L^2(\mathbf{R}^2)}. \tag{10.31}$$

Since either $\|\eta\|^2_{L^2(\mathbf{R}^2)} \leq \ell_1$ or $\|\nabla U\|^2_{L^2(\mathbf{R}^2)} \leq M_1$ and then, by (10.17), $\|\eta\|^2_{L^2(\mathbf{R}^2)} \leq \frac{M_1}{a_1^2}$, we get

$$\Big| \int_{\mathbf{R}^2} \mathcal{O}(|\eta|^3) \; dx \Big| \leq C\|\eta\|^3_{L^3(\mathbf{R}^2)} \leq C\|\eta\|_{L^2(\mathbf{R}^2)}\|\eta\|^2_{L^4(\mathbf{R}^2)} \leq C\|\eta\|^2_{L^4(\mathbf{R}^2)}. \tag{10.32}$$

Recall that $1 - \beta_* \leq \rho \leq 1 + \beta_*$ and using step 1 we find

$$\Big| \int_{\mathbf{R}^2} \rho\eta(\partial_{x_2}\theta)^2 \; dx \Big| \leq C \int_{\mathbf{R}^2} (\partial_{x_2}\theta)^2 \; dx \leq C\|\eta\|^4_{L^4(\mathbf{R}^2)}.$$

Since $c \in [0, v_s]$, from step 1 and the Cauchy-Schwarz inequality we obtain

$$\Big| \int_{\mathbf{R}^2} c\rho\eta h \, dx \Big| \leq C \|\eta\|_{L^2(\mathbf{R}^2)} \|h\|_{L^2(\mathbf{R}^2)} \leq C \|\eta\|^2_{L^4(\mathbf{R}^2)},$$

Using the definition of h, step 1 and (10.32) we now estimate

$$\begin{aligned} \Big| \int_{\mathbf{R}^2} \rho\eta(\partial_{x_1}\theta)^2 \, dx \Big| &\leq C \int_{\mathbf{R}^2} |\eta| \left(h - \frac{c}{2}\eta \right)^2 \, dx \\ &\leq C \|\eta\|_{L^\infty(\mathbf{R}^2)} \int_{\mathbf{R}^2} h^2 \, dx + C \int_{\mathbf{R}^2} |\eta|^3 \, dx \leq C \|\eta\|^2_{L^4(\mathbf{R}^2)}. \end{aligned} \tag{10.33}$$

Summing up the above estimates and using (10.30) yields the conclusion.

In steps 1 and 2 we have not used the fact that $F''(1) = 3$. Since $g(s) = \frac{v_s^2}{2} s^2 + \mathcal{O}(s^3)$ when $F''(1) = 3$, it is natural to write (10.13) in the form

$$\begin{aligned} \widehat{\eta}(\xi) = & - \mathcal{L}_c(\xi) \mathcal{F} \left(2(\partial_{x_1}\theta)^2 - \frac{v_s^2}{2}\eta^2 \right) \\ & - \mathcal{L}_c(\xi) \mathcal{F} \left(2\eta(\partial_{x_1}\theta)^2 + 2\rho^2(\partial_{x_2}\theta)^2 + 2|\nabla\rho|^2 - \left[g(\eta) - \frac{v_s^2}{2}\eta^2 \right] \right) \\ & - 2c\mathcal{L}_c(\xi) \frac{\xi_2^2}{|\xi|^2} \mathcal{F}(\eta\partial_{x_1}\theta) + 2c \frac{\xi_1\xi_2}{|\xi|^2} \mathcal{L}_c(\xi) \mathcal{F}(\eta\partial_{x_2}\theta). \end{aligned} \tag{10.34}$$

where we recall that $\mathcal{L}_c(\xi)$ is given by (10.15). We expect the term in the first line of (10.34) to be much smaller than quadratic. By the Riesz-Thorin Theorem we have $\|\eta\|_{L^4(\mathbf{R}^2)} \leq C \|\widehat{\eta}\|_{L^{4/3}(\mathbf{R}^2)}$. We will estimate the $L^{4/3}$ norm of all the terms in the right-hand side of (10.34) and we will show that they are bounded by $C\|\eta\|^2_{L^4(\mathbf{R}^2)}$.

Step 3. We have, for some constant C depending only on F,

$$\Big\| 2c \frac{\xi_1\xi_2}{|\xi|^2} \mathcal{L}_c(\xi) \mathcal{F}(\eta\partial_{x_2}\theta) \Big\|_{L^{4/3}(\mathbf{R}^2)} \leq C \|\eta\|^2_{L^4(\mathbf{R}^2)}.$$

Indeed, by the continuity of $\mathcal{F} : L^1(\mathbf{R}^2) \longrightarrow L^\infty(\mathbf{R}^2)$ and the Cauchy-Schwarz inequality one has

$$\begin{aligned} \Big\| 2c \frac{\xi_1\xi_2}{|\xi|^2} \mathcal{L}_c(\xi) \mathcal{F}(\eta\partial_{x_2}\theta) \Big\|_{L^{4/3}(\mathbf{R}^2)} & \leq C \|\mathcal{F}(\eta\partial_{x_2}\theta)\|_{L^\infty(\mathbf{R}^2)} \Big\| \frac{\xi_1\xi_2}{|\xi|^2} \mathcal{L}_c(\xi) \Big\|_{L^{4/3}(\mathbf{R}^2)} \\ & \leq C \|\eta\partial_{x_2}\theta\|_{L^1(\mathbf{R}^2)} \Big\| \frac{\xi_1\xi_2}{|\xi|^2} \mathcal{L}_c(\xi) \Big\|_{L^{4/3}(\mathbf{R}^2)} \\ & \leq C \|\eta\|_{L^2(\mathbf{R}^2)} \|\partial_{x_2}\theta\|_{L^2(\mathbf{R}^2)} \Big\| \frac{\xi_1\xi_2}{|\xi|^2} \mathcal{L}_c(\xi) \Big\|_{L^{4/3}(\mathbf{R}^2)} \\ & \leq C \|\eta\|^2_{L^4(\mathbf{R}^2)} \Big\| \frac{\xi_1\xi_2}{|\xi|^2} \mathcal{L}_c(\xi) \Big\|_{L^{4/3}(\mathbf{R}^2)}, \end{aligned}$$

where we have used the estimate $\|\partial_{x_2}\theta\|_{L^2(\mathbf{R}^2)} \leq C\|\eta\|_{L^4(\mathbf{R}^2)}$ (see Step 1) and the fact that $\|\eta\|_{L^2(\mathbf{R}^2)}$ is bounded. Thus it suffices to prove that $\Big\| \frac{\xi_1\xi_2}{|\xi|^2} \mathcal{L}_c(\xi) \Big\|_{L^{4/3}(\mathbf{R}^2)}$ is bounded independently on c.

Using polar coordinates, we find for all $q > 1$,

$$\|\mathcal{L}_c(\xi)\|^q_{L^q(\mathbf{R}^2)} = \int_{\mathbf{R}^2} \frac{|\xi|^{2q} d\xi}{(|\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2)^q} = 4 \int_0^{\pi/2} \int_0^{+\infty} \frac{r \, dr \, d\vartheta}{(r^2 + v_s^2 - c^2\cos^2\vartheta)^q}$$

$$= \tfrac{2}{q-1} \int_0^{\pi/2} \frac{d\vartheta}{(v_s^2 - c^2\cos^2\vartheta)^{q-1}} \leq \tfrac{2}{q-1} \int_0^{\pi/2} \frac{d\vartheta}{(v_s^2 - v_s^2\cos^2\vartheta)^{q-1}} = \tfrac{2}{(q-1)v_s^{2(q-1)}} \int_0^{\pi/2} \frac{d\vartheta}{(\sin\vartheta)^{2(q-1)}}.$$

Since the last integral is finite and does not depend on c if $2(q-1) < 1$, we get

$$\sup_{0\le c\le v_s} \|\mathcal{L}_c(\xi)\|_{L^q(\mathbf{R}^2)} \le C_q < \infty \qquad \text{for any } q \in \left(1, \frac{3}{2}\right). \tag{10.35}$$

In particular we have $\left\|\frac{\xi_1\xi_2}{|\xi|^2}\mathcal{L}_c(\xi)\right\|_{L^{4/3}(\mathbf{R}^2)} \le \|\mathcal{L}_c(\xi)\|_{L^{4/3}(\mathbf{R}^2)} \le C_{\frac{4}{3}}$ for $0 \le c \le v_s$ and this concludes the proof of step 3.

Step 4. There holds

$$\left\|2c\frac{\xi_2^2}{|\xi|^2}\mathcal{L}_c(\xi)\mathcal{F}(\eta\partial_{x_1}\theta)\right\|_{L^{4/3}(\mathbf{R}^2)} \le C\|\eta\|^2_{L^4(\mathbf{R}^2)}.$$

From the definition of h we have $\eta\partial_{x_1}\theta = \eta h - \frac{c\eta^2}{2}$, thus

$$\left\|2c\frac{\xi_2^2}{|\xi|^2}\mathcal{L}_c(\xi)\mathcal{F}(\eta\partial_{x_1}\theta)\right\|_{L^{4/3}(\mathbf{R}^2)} \le C\left\|\frac{\xi_2^2}{|\xi|^2}\mathcal{L}_c(\xi)\mathcal{F}(\eta^2)\right\|_{L^{4/3}(\mathbf{R}^2)} + C\|\mathcal{L}_c(\xi)\mathcal{F}(\eta h)\|_{L^{4/3}(\mathbf{R}^2)}.$$

The second term is estimated as in Step 3, using (10.35), step 1 and the fact that $\|\eta\|_{L^2(\mathbf{R}^2)}$ is bounded:

$$\|\mathcal{L}_c(\xi)\mathcal{F}(\eta h)\|_{L^{4/3}(\mathbf{R}^2)} \le \|\mathcal{L}_c(\xi)\|_{L^{4/3}(\mathbf{R}^2)}\|\mathcal{F}(\eta h)\|_{L^\infty(\mathbf{R}^2)} \le C\|\eta\|_{L^2(\mathbf{R}^2)}\|h\|_{L^2(\mathbf{R}^2)} \le C\|\eta\|^2_{L^4(\mathbf{R}^2)}.$$

For the first term we first observe that, since $c^2 \le v_s^2$,

$$\left|\frac{\xi_2^2}{|\xi|^2}\mathcal{L}_c(\xi)\right| = \frac{\xi_2^2}{|\xi|^4 + v_s^2|\xi|^2 - c^2\xi_1^2} \le \frac{\xi_2^2}{v_s^2|\xi|^2 - v_s^2\xi_1^2} = \frac{1}{v_s^2}.$$

Hence, using the estimate $\|f\|_{L^4} \le \|f\|^{\frac{2}{3}}_{L^\infty}\|f\|^{\frac{1}{3}}_{L^{4/3}}$, we get for $0 \le c \le v_s$,

$$\left\|\frac{\xi_2^2}{|\xi|^2}\mathcal{L}_c(\xi)\right\|_{L^4(\mathbf{R}^2)} \le \frac{1}{v_s^{4/3}}\left\|\frac{\xi_2^2}{|\xi|^2}\mathcal{L}_c(\xi)\right\|^{\frac{1}{3}}_{L^{4/3}(\mathbf{R}^2)} \le \frac{1}{v_s^{4/3}}\|\mathcal{L}_c(\xi)\|^{\frac{1}{3}}_{L^{4/3}(\mathbf{R}^2)} \le C.$$

(Warning: $\mathcal{L}_c$ is not uniformly bounded in $L^4(\mathbf{R}^2)$ as $c \to v_s$.) As a consequence, using the generalized Hölder inequality with $\frac{1}{4/3} = \frac{1}{4} + \frac{1}{2}$ and the Plancherel formula,

$$\left\|\frac{\xi_2^2}{|\xi|^2}\mathcal{L}_c(\xi)\mathcal{F}(\eta^2)\right\|_{L^{4/3}(\mathbf{R}^2)} \le \left\|\frac{\xi_2^2}{|\xi|^2}\mathcal{L}_c(\xi)\right\|_{L^4(\mathbf{R}^2)}\|\mathcal{F}(\eta^2)\|_{L^2(\mathbf{R}^2)} \le C\|\eta\|^2_{L^4(\mathbf{R}^2)}.$$

Combining the above estimates gives the desired conclusion.

Step 5. If $F''(1) = 3$ we have

$$\left\|\mathcal{L}_c(\xi)\mathcal{F}\left(2\eta(\partial_{x_1}\theta)^2 + 2\rho^2(\partial_{x_2}\theta)^2 + 2|\nabla\rho|^2 - \left[g(\eta) - \frac{v_s^2}{2}\eta^2\right]\right)\right\|_{L^{4/3}(\mathbf{R}^2)} \le C\|\eta\|^2_{L^4(\mathbf{R}^2)}.$$

By (10.35) and the inequality

$$\|\mathcal{L}_c(\xi)\mathcal{F}(H)\|_{L^{4/3}(\mathbf{R}^2)} \le \|\mathcal{L}_c(\xi)\|_{L^{4/3}(\mathbf{R}^2)}\|\mathcal{F}(H)\|_{L^\infty(\mathbf{R}^2)} \le C_{\frac{4}{3}}\|H\|_{L^1(\mathbf{R}^2)}$$

it suffices to estimate

$$\left\|2\eta(\partial_{x_1}\theta)^2 + 2\rho^2(\partial_{x_2}\theta)^2 + 2|\nabla\rho|^2 - \left[g(\eta) - \frac{v_s^2}{2}\eta^2\right]\right\|_{L^1(\mathbf{R}^2)}.$$

We estimate each term separately. We have already seen that $g(s) = \frac{v_s^2}{2}s^2 + \mathcal{O}(s^3)$ as $s \to 0$ because $F''(1) = 3$. By (10.31) we obtain

$$\Big\| g(\eta) - \frac{v_s^2}{2}\eta^2 \Big\|_{L^1(\mathbf{R}^2)} \leq C\|\eta\|^3_{L^3(\mathbf{R}^2)} \leq C\|\eta\|^2_{L^4(\mathbf{R}^2)}.$$

From step 2 we have

$$\Big\| |\nabla\rho|^2 \Big\|_{L^1(\mathbf{R}^2)} = \int_{\mathbf{R}^2} |\nabla\rho|^2 \, dx \leq C\|\eta\|^2_{L^4(\mathbf{R}^2)}$$

and from step 1 we get

$$\Big\| \rho^2(\partial_{x_2}\theta)^2 \Big\|_{L^1(\mathbf{R}^2)} \leq C\int_{\mathbf{R}^2} (\partial_{x_2}\theta)^2 \, dx \leq C\|\eta\|^4_{L^4(\mathbf{R}^2)} \leq C\|\eta\|^2_{L^4(\mathbf{R}^2)}.$$

Finally, as in (10.33) we infer that

$$\Big\| \eta(\partial_{x_1}\theta)^2 \Big\|_{L^1(\mathbf{R}^2)} \leq C\|\eta\|^2_{L^4(\mathbf{R}^2)}.$$

Gathering the above estimates we get the conclusion.

Step 6. The following estimate holds:

$$\Big\| \mathcal{L}_c(\xi)\mathcal{F}\left(2(\partial_{x_1}\theta)^2 - \frac{v_s^2}{2}\eta^2\right) \Big\|_{L^{4/3}(\mathbf{R}^2)} \leq C\|\eta\|^2_{L^4(\mathbf{R}^2)}.$$

Indeed, arguing as is step 5 and using the definition of h, the Cauchy-Schwarz inequality and step 1 we deduce

$$\Big\| \mathcal{L}_c(\xi)\mathcal{F}\left(2(\partial_{x_1}\theta)^2 - \frac{v_s^2}{2}\eta^2\right) \Big\|_{L^{4/3}(\mathbf{R}^2)} \leq C_{\frac{4}{3}}\Big\| 2\left(h - \frac{c}{2}\eta\right)^2 - \frac{v_s^2}{2}\eta^2 \Big\|_{L^1(\mathbf{R}^2)}$$

$$\leq C\|h^2\|_{L^1(\mathbf{R}^2)} + C\|\eta\|_{L^2(\mathbf{R}^2)}\|h\|_{L^2(\mathbf{R}^2)} + \frac{v_s^2 - c^2}{2}\int_{\mathbf{R}^2} \eta^2 \, dx \leq C\|\eta\|^2_{L^4(\mathbf{R}^2)}.$$

Step 7. Conclusion.

Using the Riesz-Thorin theorem, we have $\|\eta\|_{L^4(\mathbf{R}^2)} \leq C\|\hat{\eta}\|_{L^{4/3}(\mathbf{R}^2)}$. Coming back to (10.34) and gathering the estimates in steps 3-6, we deduce

$$\|\eta\|_{L^4(\mathbf{R}^2)} \leq C\|\hat{\eta}\|_{L^{4/3}(\mathbf{R}^2)} \leq C\|\eta\|^2_{L^4(\mathbf{R}^2)},$$

where C depends only on F. Consequently, either $\|\eta\|_{L^4(\mathbf{R}^2)} = 0$, or there is a constant $\kappa > 0$ such that $\|\eta\|_{L^4(\mathbf{R}^2)} \geq \kappa$. If $\|\eta\|_{L^4(\mathbf{R}^2)} = 0$ we have $\eta = 0$ a.e. and from (10.17) we get $\|\nabla U\|_{L^2(\mathbf{R}^2)} = 0$, hence U is constant. If $\|\eta\|_{L^4(\mathbf{R}^2)} \geq \kappa$, (10.28) implies that there are $\ell_* > 0$ and $k_* > 0$ such that $\|\eta\|_{L^2(\mathbf{R}^2)} \geq \ell_*$ and $\|\nabla U\|_{L^2(\mathbf{R}^2)} \geq k_*$. The proof of Proposition 1.4 is complete. □

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